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REPORT No. 961

An Analytical Treatment Of The Problem Of Triangulation By Stereophotogrammetry

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DEPARTMENT OF THE ARMY PROJECT No. 5B0306011
ORDNANCE RESEARCH AND DEVELOPMENT PROJECT No. TB3-0538

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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AN ANALYTICAL TREATMENT of the PROBLEM of TRIANGULATION by STEREOPHOTOGRAMMETRY

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TABLE of CONTENTS

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ABSTRACT

An analytical solution for the basic problem of triangulation by stereophotogrammetry is derived. The most general case is defined as the problem of determining simultaneously the orientations of the two cameras, whereby all 18 unknown parameters of the orientation are considered with no limitations on the camera orientations. The process of triangulation is treated as the computation of spatial coordinates as functions of the elements of orientation and the corresponding plate measurements. The least squares solution derived is based on rigorous mathematical expressions which connect the plate measurements with the unknown parameters. In contrast to the conventional approach, the separation of the orientation problem into the two phases of relative and absolute orientation is avoided. This analytical solution can be based on a few basic theorems of solid analytical geometry. Because the observation method - monocular or stereoscopic - does not influence the formulas expressing the rigorous geometry, it is possible to make use of absolute control points which are not common to the area covered by the two photographs under consideration. Thus more favorable geometry is introduced into the problem of the double-point intersection in space.

The least squares solution derived is suitable for any number and any combination of absolute, partially absolute and relative control points. In addition, any one of the elements of orientation - including the base line components - may be enforced in the solution. By applying the concept of pseudo-residuals and by introducing cross-weights, it is possible to treat the least squares solution like a problem involving independent indirect measurements. Furthermore, it is shown that the normal equation system can be formed step by step. This method has merit when electronic computers are used since the number of points carried in the solution has only a slight effect on the amount of memory space needed.

The introduction of rotational auxiliaries which are essentially direction cosines and the combination of these with the plate coordinates as linear

auxiliaries render the coefficients of the observational equations as partial differential quotients in terms attractive for an analytical treatment. The process of triangulation is treated as a part of the process of orientation as well as an independent computational procedure.

A special chapter deals with the determination of the mean errors of the observations, of the elements of orientation, and of the triangulation results.

Finally, the application of the proposed analytical method to the problem of control extension is discussed in principle. It is shown that the method used on the universal plotters for a strip triangulation procedure is an approximate solution only, because it is based on incomplete conditional equations.

The rigorous geometry for the problem of extension is interpreted as the condition that rays originating from three consecutive camera stations have to intersect for at least one point located in the area common to the three photographs under consideration. The corresponding conditional equations are derived and the coefficients of the corresponding observational equations are given. It is shown that it is now possible to include in the extension, models which are formed by the combination of photographs taken at every other camera position. The thus extended base line provides for a favorable base-height ratio otherwise obtainable only by convergent photography.

As stated before, the final normal equation system can be formed step by step. Attention is called to the fact that the matrix of the unknown parameters is filled in the neighborhood of the diagonal only, thus making it possible to use an iterative subroutine.

The method presented for a strip triangulation is useful in a least squares treatment of a block triangulation also.

I. INTRODUCTION

Photogrammetry may be defined as the science in which geometrical properties of objects are analyzed in quantitative terms from their images recorded on photographs. Stereophotogrammetry, in particular, deals with the triangulation of pencils of rays originating from two camera stations by applying the technique of stereoscopic observation.

The process of triangulating two corresponding pencils of rays consists of the restitution of the orientation of the two photographs under consideration and the reconstruction of the model. The analytical equivalent of the restitution of the orientation is the determination of two sets of nine degrees of freedom, namely, six translations and three rotations for each of the two camera stations. Expressed analytically, the reconstruction of the model is the process of triangulating individual points whereby each triangulation is based on the 18 elements of orientation and four corresponding plate measurements.

Approaching the problem analytically, for the general case the 18 elements of orientation are obtained from a least squares adjustment based on a sufficient number of conditions of intersection (relative orientation) and on independently established control points (absolute orientation). Thus the general photogrammetric problem is essentially an interpolation procedure by which systematic errors may be eliminated and the propagation of residual errors in the final triangulation procedure decreased. Depending upon the arrangement of the triangulating pencils of rays (normal case or oblique case) and upon the type of photogrammetric instrumentation used (aerial cameras or phototheodolites), the number of independently given orientation elements will vary. However, this fact does not eliminate the basic necessity of computing the remaining orientation elements by a least squares adjustment as stated above. Thus, it appears to be unnecessary to distinguish between different types of photogrammetry because the methods of stereophotogrammetry applied to all measuring problems, e.g., in geodetic and industrial work, are based on essentially the same geometrical and physical principles.

A general theory for the error propagation deals with the mean error of an observation of unit weight, with the errors of the elements of orientation and their propagation into the mean errors of the spatial coordinates of the model.

II. GENERAL REMARKS

The concept of measuring should always include the principle of overdetermination. Therefore, a final determination of the unknown parameters should be the result of an adjustment. The significance of such an answer is determined by the computed estimates of the precision. In general, it is possible to obtain approximate values for the unknown parameters without difficulty. Consequently, in the process of adjustment, corrections to the approximate values must be computed to render the final result. The process of a rigorous adjustment requires that the weighted sum of the squares of the residuals of the original measurements be a minimum. If the measuring errors are normally distributed, the result represents the most probable values for the unknown parameters. The precision of the result is computed from the weight coefficients obtained from the adjustment and the mean error of unit weight, which may be either independently determined or computed from the sum of the squares of the residuals. The specific least squares adjustment is given by rigorous mathematical expressions which connect the original measurements with the unknown parameters. If these functions are not linear, it is convenient to linearize them by applying the Taylor series, neglecting second and higher order terms. In such cases it may be necessary to compute a series of corrections by an iterative process in order to achieve a desired accuracy in the final result.

In an analytical treatment of any triangulation problem - including photogrammetric problems - the above mentioned procedure must be considered as a part of a least squares adjustment.

Before analyzing our problem in detail, let us state the objective. The sole purpose in topographic photogrammetry, and in numerous non-topographic applications of photogrammetric measuring methods, is the reconstruction of the object photographed. In accordance with such an objective the efforts in research and practical application are often limited to problems of production and quality of the stereoscopic model and of its evaluation. In such cases, a limited amount of emphasis is put on the geometry and the physical conditions existing during the process of taking the photographs. In various non-topographic applications, however, photographs are taken either to calibrate certain physical parameters of the cameras themselves or to determine absolute values of the orientation elements[1]. Typical examples are: a) the determination of position and attitude of an airborne camera from photographed ground control points for the purpose of trajectory determination of airplanes or guided missiles,[2] and b) the calibration and orientation of photographs taken with ground based cameras for the purpose of providing a reference datum, where such cameras are mounted rigidly in relation to other measuring equipment. In addition, a rigorous mathematical treatment of the problem of triangulation is necessary for a study of the propagation of random and systematic errors in both single models and extensions.

In general, an approximate solution, though useful for a specific case, will not necessarily be adaptable to other cases. However, a general and rigorous analytical treatment of the simultaneous orientation of two photogrammetric cameras will be applicable to any measuring method based on the stereophotogrammetric principle. Approximate solutions which may be desirable for economic reasons in special cases can be derived from a general solution by introducing certain assumptions peculiar to such cases. Therefore an analytical treatment should be sufficiently general as to the least squares procedure and the code for electronic computers to permit the use of the method for a wide range of stereophotogrammetric measuring problems.

III. THE GEOMETRICAL ANALYSIS

In making a rigorous geometrical analysis of the simultaneous orientation of two photogrammetric cameras, we shall first consider the method of deriving the orientation used in any optical-mechanical stereoscopic restitution instrument. The spatial position and orientation of a camera with known optical characteristics are defined by six parameters, namely, three translations and three rotations. Consequently, for the two cameras under consideration we are dealing with 12 unknown parameters.

Two bundles of projective rays can be oriented with respect to each other

[1] cf. [8]. (References at the end of the paper)
[2] cf. [9].

from measurements made on the photographs without reference to absolute control data. Five pairs of corresponding rays are necessary and sufficient to establish this relative orientation which is equivalent to the construction of a true model of the object photographed by the two cameras. In the process of determining the remaining seven unknowns we must find the scale and the absolute orientation of the model. The latter establishes the model with respect to a given control system by three additional rotations and translations which are equivalent to a coordinate transformation.

The separation of the orientation problem into the two fundamental processes of relative and absolute orientation has been predominant in practical photogrammetry and, consequently, has greatly influenced the related error theory. The separation of the two processes of orientation has been strictly based upon the practical methods of restitution. From the standpoint of both practice and error theory, this separation has its disadvantages. The model obtained from the relative orientation - even if it were flawless - is of interest in very special cases only. The relative model usually has to be transformed into an absolute one. During this process absolute control in excess is often used to improve the preliminary relative orientation (compensation for model deformations) as well as to minimize the influence of existing tensions within the given control. In any case the absolute control must be considered to be of greater importance than the relative information.

In an analytical treatment the absolute orientations of the two projective bundles of rays under consideration are functions of the positions of photographed images on the plates. This conception automatically includes the relative as well as the absolute orientation of the model. In addition, the sum of the squares of the weighted residuals of the original coordinate measurements on the photographs must be minimized. Therefore, the orientation problem may be expressed by formulas explicit in terms of the measured plate coordinates and preferably so arranged that a minimum number of measurements and residuals appears in each of the observational equations.

It is of further interest to note that in an analytical treatment the method of observation - monocular or stereoscopic - does not influence the formulas expressing the rigorous geometry. In other words, monocularly and/or stereoscopically observed coordinates may be introduced into the computation so long as the condition is satisfied that the sum of the squares of the weighted residuals of the original measurements is minimized. Consequently, an analytical method may use absolute control data outside the field common to both photographs. Thus an extended basal area is obtained for the double point resection in space, a fact which will make it possible to use the favorable geometry which is characteristic of the single spatial resection[3].

In order to obtain a general solution for our problem, the least squares adjustment must be suitable for all numbers and types of reference points and their corresponding plate measurements - provided that the information is sufficient for a unique solution and satisfies the principle of the relative orientation. The three types of reference points are: 1) absolute control points given by their spatial coordinates, 2) partially absolute control points given either by their Z coordinates or by their X and Y coordinates, and 3) relative control points. Points of the first type are not necessarily restricted to the area common to both photographs but the two latter types of points must lie within this area.

3 cf. [9]

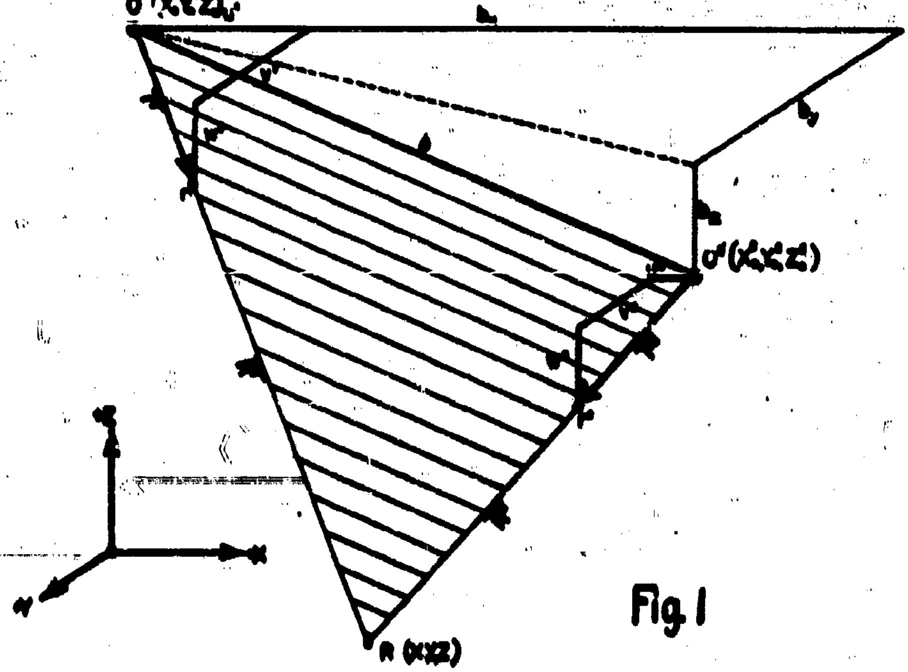


Fig. I

The basic geometry necessary to meet the above mentioned requirements is simple. Assuming a unique solution, we see from Figure 1 that the condition for correct orientation with respect to an absolute control point R, given by X, Y, Z, is equivalent to the condition that the center of projection O, the image point r and the control point R are collinear. In case the absolute control point is located within the area common to both photographs, the condition of collinearity is valid for both the left and right photographs. In such a case, the condition of intersection of two corresponding rays - namely at point R - is automatically satisfied. The fact that the left and right cameras are fully independent in regard to the condition of collinearity simplifies the least squares adjustment. Furthermore, this fact is interesting from the standpoint of error theory. A similar geometrical condition may be obtained for relative control points. Also from Figure 1 we see that the condition that two corresponding rays intersect is equivalent to the condition that the two centers of projection O' and O", together with two corresponding image points r' and r", are coplanar. In deriving the necessary formulas we shall see that the two geometrical conditions mentioned above are adequate for the solution of our problem.

From Figure 1 we obtain:

$$\vec{R}' = \mu'\vec{r}' \quad \text{and} \quad \vec{R}'' = \mu''\vec{r}'' \tag{1}$$

where μ' and μ'' are scale factors. The prime and double prime indices refer to left and right stations, respectively.

Correspondingly :

$$\begin{aligned} X &= X_o' + \mu'u' = X_o'' + \mu''u'' \\ Y &= Y_o' + \mu'v' = Y_o'' + \mu''v'' \\ Z &= Z_o' + \mu'w' = Z_o'' + \mu''w'' \end{aligned} \tag{2}$$

Each of the triplets of formulas (2) is the analytical expression for the condition that the points O', r', R and O", r", R, respectively, lie on straight lines. Because the point R is common to both triplets, these lines satisfy in addition the condition of intersection at the point R.

From formula (2) follows:

$$u' = \frac{(X)'}{\mu'} \qquad u'' = \frac{(X)''}{\mu''} \qquad X - X_o = (X)$$
$$v' = \frac{(Y)'}{\mu'} \quad \text{and} \quad v'' = \frac{(Y)''}{\mu''} \quad \text{where} \quad Y - Y_o = (Y) \tag{3}$$
$$w' = \frac{(Z)'}{\mu'} \qquad w'' = \frac{(Z)''}{\mu''} \qquad Z - Z_o = (Z)$$

From Figure 2 we read

$$\begin{aligned} \vec{r} &= iu + jv + kw = \hat{i}\bar{x} + \hat{j}\bar{y} + \hat{k}c \\ \vec{R} &= i(X) + j(Y) + k(Z) \end{aligned} \tag{4}$$

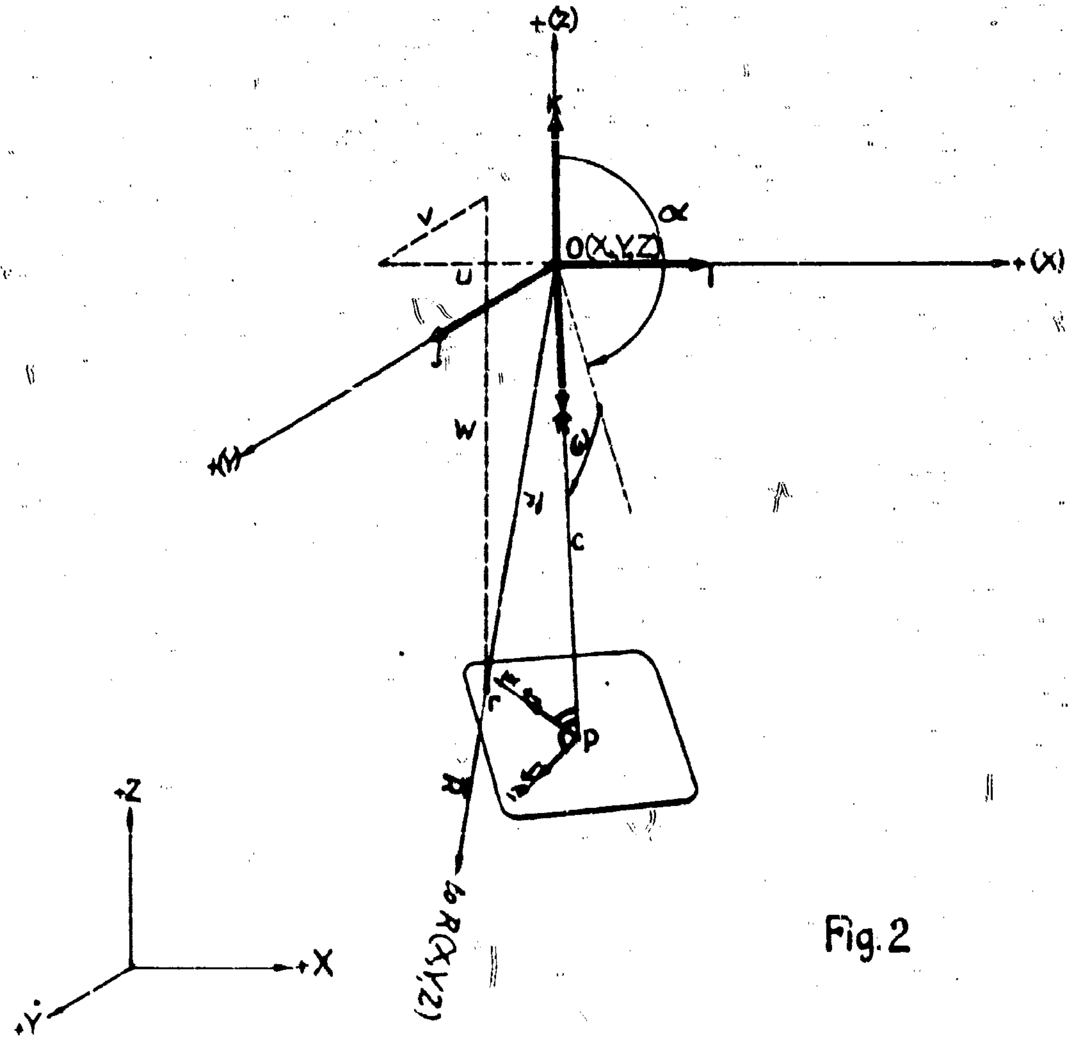


Fig. 2

Further we obtain $\vec{r}\,\hat{k} = c$ and by formulas (1) $\mu\,\vec{r}\,\hat{k} = \vec{R}\,\hat{k}$ and, therefore,

$$\mu = \frac{\vec{R}\,\hat{k}}{c} \tag{5}$$

With the transformation matrix of the two vector triplets

$$\begin{array}{c|c} & \begin{matrix} \hat{i} & \hat{j} & \hat{k} \end{matrix} \\ \hline \begin{matrix} i \\ j \\ k \end{matrix} & \begin{pmatrix} i\hat{i} & i\hat{j} & i\hat{k} \\ j\hat{i} & j\hat{j} & j\hat{k} \\ k\hat{i} & k\hat{j} & k\hat{k} \end{pmatrix} \end{array}$$

and from formulas (4), we obtain

$$\mu = \frac{(X) \sin a \cos \omega + (Y) \sin \omega + (Z) \cos a \cos \omega}{c} \tag{6}$$

and

$$\begin{aligned} u &= \bar{x} \cos a - \bar{y} \sin a \sin \omega + c \sin a \cos \omega \\ v &= \bar{y} \cos \omega + c \sin \omega \\ w &= -\bar{x} \sin a - \bar{y} \cos a \sin \omega + c \cos a \cos \omega \end{aligned} \tag{7}$$

In formula (7) we have from Figure 3

$$\begin{aligned} \bar{x} &= -(x - x_p) \cos \varkappa - (y - y_p) \sin \varkappa \\ \bar{y} &= -(x - x_p) \sin \varkappa + (y - y_p) \cos \varkappa \end{aligned} \tag{8}$$

We now introduce auxiliaries which are used throughout the report.

$$\begin{aligned} A_1 &= -\cos a \cos \varkappa + \sin a \sin \omega \sin \varkappa & A_2 &= -\cos a \sin \varkappa - \sin a \sin \omega \cos \varkappa \\ B_1 &= -\cos \omega \sin \varkappa & B_2 &= \cos \omega \cos \varkappa \\ C_1 &= \sin a \cos \varkappa + \cos a \sin \omega \sin \varkappa & C_2 &= \sin a \sin \varkappa - \cos a \sin \omega \cos \varkappa \\ D_1 &= \sin a \cos \omega & D_2 &= \sin a \sin \omega \\ E_1 &= \sin \omega & E_2 &= \cos \omega \\ F_1 &= \cos a \cos \omega & F_2 &= \cos a \sin \omega \\ G_1 &= \sin a & G_2 &= \cos a \\ H_1 &= \sin \varkappa & H_2 &= \cos \varkappa \end{aligned} \tag{9}$$

xy-plane as diapositive seen from O

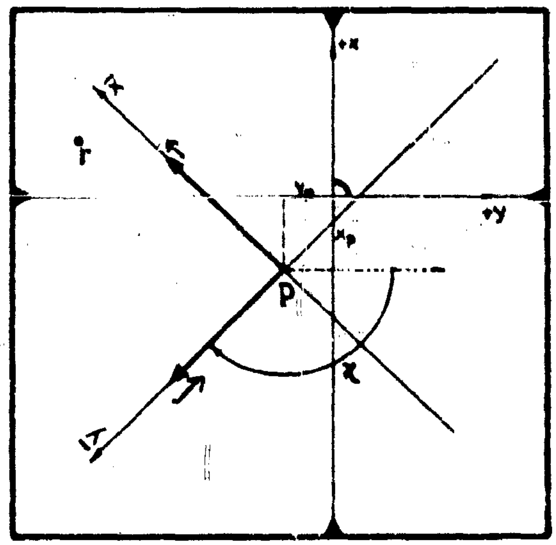


Fig. 3

By substituting (9) into (6)

$$\mu = \frac{(X)D_1 + (Y)E_1 + (Z)F_1}{c} \tag{10}$$

and by substituting (8) into (7), using the expressions (9)

$$\begin{aligned} u &= (x - x_p)A_1 + (y - y_p)A_2 + cD_1 \\ v &= (x - x_p)B_1 + (y - y_p)B_2 + cE_1 \\ w &= (x - x_p)C_1 + (y - y_p)C_2 + cF_1 \end{aligned} \tag{11}$$

Substituting formulas (10) and (11) into (2) and (3), and using

$$X = \frac{(Z)u}{w} + X_o \qquad \text{and} \qquad Y = \frac{(Z)v}{w} + Y_o \tag{12}$$

we have for the left and right stations:

$$\begin{aligned} X &= \frac{(Z-Z_o') \left[(x'-x_p')A_1' + (y'-y_p')A_2' + c'D_1'\right]}{(x'-x_p')C_1' + (y'-y_p')C_2' + c'F_1'} + X_o' \\ &= \frac{(Z-Z_o'') \left[(x''-x_p'')A_1'' + (y''-y_p'')A_2'' + c''D_1''\right]}{(x''-x_p'')C_1'' + (y''-y_p'')C_2'' + c''F_1''} + X_o'' \\ Y &= \frac{(Z-Z_o') \left[(x'-x_p')B_1' + (y'-y_p')B_2' + c'E_1'\right]}{(x'-x_p')C_1' + (y'-y_p')C_2' + c'F_1'} + Y_o' \\ &= \frac{(Z-Z_o'') \left[(x''-x_p'')B_1'' + (y''-y_p'')B_2'' + c''E_1''\right]}{(x''-x_p'')C_1'' + (y''-y_p'')C_2'' + c''F_1''} + Y_o'' \end{aligned} \tag{13}$$

and

$$\begin{aligned} x' &= \frac{c' \left[(X-X_o')A_1' + (Y-Y_o')B_1' + (Z-Z_o')C_1'\right]}{q'} + x_p' \\ y' &= \frac{c' \left[(X-X_o')A_2' + (Y-Y_o')B_2' + (Z-Z_o')C_2'\right]}{q'} + y_p' \end{aligned}$$

where

$$q' = (X-X_o')D_1' + (Y-Y_o')E_1' + (Z-Z_o')F_1' \tag{14}$$

and

$$x'' = \frac{c''\left[(X-X''_o)A''_1 + (Y-Y''_o)B''_1 + (Z-Z''_o)C''_1\right]}{q''} + x''_p$$

(14 cont.)

$$y'' = \frac{c''\left[(X-X''_o)A''_2 + (Y-Y''_o)B''_2 + (Z-Z''_o)C''_2\right]}{q''} + y''_p$$

where

$$q'' = (X-X''_o)D''_1 + (Y-Y''_o)E''_1 + (Z-Z''_o)F''_1$$

A more detailed derivation of the formulas (13) and (14) may be found in [8][4].

The coordinates of R may be only partially given, either with its elevation Z or with its location X and Y. We obtain from the three equalities in formula (2), in case Z is given

$$Z = \frac{Z'_o u'w'' - Z''_o u''w' + b_x w'w''}{u'w'' - u''w'} = \frac{Z'_o v'w'' - Z''_o v''w' + b_y w'w''}{v'w'' - v''w'} \tag{15}$$

and in case X and Y are given

$$X = \frac{X''_o u'w'' - X'_o u''w' - b_z u'u''}{u'w'' - u''w'}$$

$$Y = \frac{Y''_o v'w'' - Y'_o v''w' - b_z v'v''}{v'w'' - v''w'} \tag{16}$$

In formulas (15) and (16) we have introduced the base line components

$$b_x = X''_o - X'_o$$

$$b_y = Y''_o - Y'_o \tag{17}$$

$$b_z = Z''_o - Z'_o$$

Formulas (15) are obtained by equalizing the X and Y coordinates of point R from the left and right stations. Thus, for a given value of Z, the condition of intersection of the two corresponding rays is satisfied. Similarly, formulas (16) are obtained by equalizing the Z coordinate of point R from the left and right station. If we introduce the given values for the X and Y coordinates, the condition of intersection is again satisfied. Each equation in formulas (15) and (16) may be written as a conditional equation. From formula (15) we obtain:

$$(Z)''u''w' - (Z)'u'w'' + b_x w'w'' = 0$$

and (18)

$$(Z)''v''w' - (Z)'v'w'' + b_y w'w'' = 0$$

From formulas (16) we derive:

$$(X)'u''w' - (X)''u'w'' - b_z u'u'' = 0$$

and (19)

$$(Y)'v''w' - (Y)''v'w'' - b_z v'v'' = 0$$

Finally, we shall consider the case in which the absolute position of point R is unknown and only the condition of intersection is given. From the equality in formula (15) we obtain:

$$b_x(v'w'' - v''w') + b_y(u''w' - u'w'') + b_z(u'v'' - u''v') = 0 \tag{20}$$

This equation may be directly obtained by a different approach.[5] The condition of intersection for any two corresponding rays is satisfied if the two rays lie in a plane (basal plane) with the base line O'O". (See Figure 1). This condition is equivalent to the condition that the four points O', O", r' and r" are coplanar. The coplanarity of these four points is satisfied if the determinant formed by the spatial coordinates vanishes. With respect to the (X), (Y), (Z) - system we have:

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ b_x & b_y & b_z & 1 \\ u' & v' & w' & 1 \\ b_x + u'' & b_y + v'' & b_z + w'' & 1 \end{vmatrix} = 0$$

This determinant is identical with the conditional equation (20). Formula (20) represents the basic expression for the relative orientation for the general case. If both b_y and b_z are equal to zero, we have:

$$\frac{v'}{w'} = \frac{v''}{w''} \tag{21}$$

or from formulas (12)

$$\frac{(Y)'}{(Z)'} = \frac{(Y)''}{(Z)''} \tag{22}$$

For the case in which $b_z = 0$, the elevations of the stations Z_0' and Z_0'' are equal, and consequently $(Z)' = (Z)''$. Thus we may write $(Y)' = (Y)''$, which is the traditional presentation for the condition of relative orientation as given by v. Gruber.[6] The condition of equality for the (Y) coordinates is used in the process of relative orientation of independent pairs of photographs in the optical-mechanical restitution equipment. Actually, the observation of any (Y) coordinate makes it necessary to introduce a corresponding (Z) coordinate. Consequently, the process of relative orientation is actually based on formula (22) which may be written as:

5 cf. [10]

6 cf. [3] pp. 27 - 29

$$\tan \lambda' = \tan \lambda'' \qquad (23)$$

λ is the spatial angle formed by an arbitrarily chosen reference plane containing the base line and a second plane containing the base line and the corresponding target ray. Thus, λ is the polar bearing of R. Equation (23) states that the two polar bearings of R as obtained from the left and right stations must be equal, a requirement which is identical with the above mentioned general condition that the two corresponding rays, with the base line, lie in a plane. In this connection, it may be mentioned that e.g. in Finsterwalder's[7] formula (10), given as the basic formula for the relative orientation, the first quotient equals $\tan \lambda''$ and the second quotient equals $\tan \lambda'$. Thus, his formula represents the special case of the relative orientation as expressed by our formulas (23).

Summarising the results obtained so far, we see that:

a) Each absolute control point, defined by its X,Y,Z coordinates, gives rise to two independent equations for each nodal point position from which the control point is photographed. Thus, if the absolute control point lies in the area common to both photographs, we obtain four independent equations and in addition satisfy the condition that the two corresponding rays intersect. (Formulas (14)).

b) Each partially absolute control point, given by either its Z coordinate or its X and Y coordinates, gives rise to two independent equations which satisfy the condition of intersection of two corresponding rays. (Formulas (18) or (19), respectively).

c) Each relative control point gives rise to one independent equation and represents the condition that two corresponding rays intersect. (Formula (20)).

From the results given in (a) to (c) above we conclude that for a unique solution it is necessary and sufficient to have, for example, for the determination of twelve unknowns: a) two absolute control points which are situated in the area common to both photographs giving rise to eight equations and satisfying two conditions of intersection; b) one partially absolute control point giving rise to two additional equations and increasing the number of enforced intersections to three; c) in addition two relative control points increasing the number of equations to the necessary twelve and establishing the fourth and fifth intersections of corresponding pairs of rays.

These requirements are identical with those necessary and sufficient to solve the problem in the conventional way, which consists of dividing the orientation problem into the phases of relative and absolute orientation as stated at the beginning of this chapter.

In applied aerial photogrammetry the difference in flying height between two successive photographs - the base component b_z - is often determined independently by Statoscope registration, and experience has shown that it is sometimes desirable to enforce this measurement in the process of orientation. Analytically speaking, there exists in such a case an additional conditional equation of the form

7 cf. [4], p. [illegible]

$$Z_o'' - Z_o' - b_z = 0$$

Because of the simplicity of this formula it seems advisable to write

$$Z_o'' = Z_o' + b_z \tag{24}$$

and substitute this expression into the formulas (14), (18), (19), and (20). Similarly, the same procedure may be followed for the b_x and b_y components. The formulas introducing the unknowns b_x, b_y and b_z are given below.

$$x' = \frac{c'\left[(X)'A_1' + (Y)'B_1' + (Z)'C_1'\right]}{q'} + x_p'$$

$$y' = \frac{c'\left[(X)'A_2' + (Y)'B_2' + (Z)'C_2'\right]}{q'} + y_p'$$

where

$$q' = (X)'D_1' + (Y)'E_1' + (Z)'F_1'$$

and

$$x'' = \frac{c''\left\{\left[(X)' - b_x\right]A_1'' + \left[(Y)' - b_y\right]B_1'' + \left[(Z)' - b_z\right]C_1''\right\}}{q''} + x_p'' \tag{14^a}$$

$$y'' = \frac{c''\left\{\left[(X)' - b_x\right]A_2'' + \left[(Y)' - b_y\right]B_2'' + \left[(Z)' - b_z\right]C_2''\right\}}{q''} + y_p''$$

where $q'' = \left[(X)' - b_x\right]D_1'' + \left[(Y)' - b_y\right]E_1'' + \left[(Z)' - b_z\right]F_1''$

and

$$\left.\begin{aligned} &(Z)'(u''w' - u'w'') - b_z u''w' + b_x w'w'' = 0 \\ &(Z)'(v''w' - v'w'') - b_z v''w' + b_y w'w'' = 0 \end{aligned}\right\} \tag{18^a}$$

$$\left.\begin{aligned} &(X)'(u''w' - u'w'') + b_x u'w'' - b_z u'u'' = 0 \\ &(Y)'(v''w' - v'w'') + b_y v'w'' - b_z v'v'' = 0 \end{aligned}\right\} \tag{19^a}$$

Formula (20) remains unchanged.

In order to enforce any or all base line components in the orientation process the corresponding unknown parameter corrections need only to be eliminated from the computation.

IV. THE LEAST SQUARES ADJUSTMENT

A. The process of orientation

1) The least squares solution

Formulas (14), (18), (19) and (20) express the functional relationship between the plate coordinates and the given control data for the process of orienting two photogrammetric cameras simultaneously. Consequently, these formulas may be considered as the basis for a rigorous least squares adjustment. In formulas (18), (19) and (20) more than one observation is involved, leading to more than one residual. Moreover these observations together with the unknown parameters of the solution must satisfy certain conditions. Therefore we must deal with the general problem of a least squares adjustment. A complete treatment of this problem was first prepared by Helmert[8].

The form of the linearised conditional equations for the different types of control points is shown in the observational equations (25). The actual values of the coefficients will be dealt with in a later chapter[9].

Formulas (25) show that for absolute control points only one observation appears in each equation and no observation appears in more than one observational equation. Hence, the coefficient matrix of the residuals for these equations is diagonal and, in fact, is equal to the unit matrix. Furthermore, the conditional equations for each station have only those coefficients which correspond to the elements of orientation of their respective stations. As already mentioned on page 10, for a case in which no partially absolute and no relative control points are present, the stereo-orientation will reduce to two independent camera orientations.

In the equations arising from partially absolute control points, four observations appear in each equation. However, the same observations appear in only two equations. The coefficient matrix of the residuals is therefore a diagonally arranged sequence of non-overlapping 2x4 submatrices.

Finally, in the equations arising from relative control points, four observations appear in each equation and none of these appears in any other equation. Thus, the coefficient matrix of the residuals is a diagonally arranged sequence of non-overlapping 1x4 submatrices.

For the partially absolute and relative control points the elements of orientation for both stations appear in the corresponding conditional equations and, consequently, the coefficient matrix of the parameters is filled.

8 cf. [5] pp. 215 - 222

9 cf. pp. 29 - 34

No.	Type of Control Point	Residuals v	Unknowns (Δ's)	Absolute Terms −L	
	[illegible]		$-A_x - B_x - C_x + D_x - E_x - F_x - G_x - 1$	$+\Delta i_x$	$= 0$
	[illegible]		$-A_y - B_y - C_y + D_y - E_y - F_y - G_y - 1$	$+\Delta i_y$	$= 0$
	[illegible]		$-A_x'' - B_x'' - C_x'' - D_x'' - E_x'' - F_x'' - G_x'' - 1$	$+\Delta i_x''$	$= 0$
	[illegible]		$-A_y'' - B_y'' - C_y'' - D_y'' - E_y'' - F_y'' + G_y'' - 1$	$+\Delta i_y''$	$= 0$
	[illegible]	$a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$	$-(A)_1 - (B)_1 - (C)_1 - (D)_1 + (F)_1 - (G)_1 - (a)_1 - (a)_2 - (J)_1 - (K)_1 - (L)_1 - (D) + (O)_1 - (P)_1 - (a)_3 - (a)_4$	$-(\Delta)_1$	$= 0$
	Z	$b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4$	$-(A)_2 - (B)_2 - (C)_2 - (D) - (F)_2 + (G)_2 - (b)_1 - (b)_2 - (J)_2 + (K)_2 - (L)_2 - (D) - (O)_2 - (P)_2 - (b)_3 - (b)_4$	$-(\Delta)_2$	$= 0$
4	[illegible]	$a_{x_1} v_1 - a_{x_2} v_2 + a_{x_3} v_3 + a_{x_4} v_4$	$+(A)_X + (B)_X - (C)_X - (D)_X + (F)_X + (G)_X - (a)_{X_1} - (a)_{X_2} - (J)_X + (K)_X - (L)_X - (M)_X - (F)_X - (P)_X - (a)_{X_3} - (a)_{X_4}$	$-(\Delta)_X$	$= 0$
	X, Y	$a_{y_1} v_1 - a_{y_2} v_2 + a_{y_3} v_3 + a_{y_4} v_4$	$-(A)_Y - (B)_Y - (C)_Y - (E)_Y - (F)_Y - (G)_Y - (a)_{Y_1} - (a)_{Y_2} - (J)_Y - (K)_Y - (L)_Y - (M)_Y - (F)_Y - (P)_Y - (a)_{Y_3} - (a)_{Y_4}$	$-(\Delta)_Y$	$= 0$
5	Relative	$(a)_1 v_1 - (a)_2 v_2 - (a)_3 v_3 - (a)_4 v_4$	$-(A) - (B) - (C) - (D) - (E) - (F) - (G) - (a)_1 - (a)_2 - (J) - (K) - (L) - (D) - (E) - (F) - (P) - (a)_3 - (a)_4$	$-(\Delta)$	$= 0$

formulas(25)

Using matrix notation[10], formulas (25) may be written as:

Group	Observational equations for
1	Absolute Control Points for the left station
2	Absolute Control Points for the right station
3	Partially Absolute Control Points with Z given
4	Partially Absolute Control Points with X and Y given
5	Relative Control Points

$$\begin{bmatrix} A_1 & 0 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 \\ 0 & 0 & 0 & A_4 & 0 \\ 0 & 0 & 0 & 0 & A_5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix} \Delta - \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix} = 0 \tag{26}$$

where the A_i are the coefficient matrices of the corresponding residual vectors v_i. In particular $A_1 = A_2 = I$, the unit matrix. The B_i are the coefficient matrices of the vector of the parameter corrections Δ. The L_i are the vectors of the absolute terms of the conditional equations. The indices 1 to 5 are chosen to specify the corresponding group of observational equations.

We may rewrite the system of observational equations (26) with obvious notation as

$$Av + B\Delta - L = 0 \tag{27}$$

10 cf. e.g. [2] and [7]

Assuming the observations to be independent and normally distributed, the most probable values are obtained by minimizing $v^T P v$, where P denotes the weight matrix

$$P = \begin{bmatrix} p_1 & 0 & \cdot & \cdot & 0 \\ 0 & p_2 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & p_n \end{bmatrix} \tag{28}$$

where the p's are the weights of the observations. The introduction of weighting factors may become necessary to express various degrees of precision with which the original observations have been obtained due to the method of measuring or due to a varying image quality caused e.g., by loss of definition towards the edges of the photograph. In case the observations all have the same weight, it is convenient to consider P as the unit matrix.

A direct solution of our problem is obtained, as Helmert has shown, by minimizing the following function:

$$v^T P v - 2 k^T (A v + B \Delta - L) \tag{29}$$

where k denotes a vector of unknown Lagrange multipliers or correlates.

Setting each of the differential quotients for $v_1, v_2, \ldots v_n$, $\Delta_1, \Delta_2, \ldots \Delta_u$ equal to zero, we obtain (n + u) equations which, together with the previously mentioned r conditional equations (formulas (26)), are necessary and sufficient for the determination of the n residuals (v), the u unknown parameter corrections (Δ) and the r correlates (k). The differentiation for $v_1, v_2, \ldots v_n$ gives:

$$P v - A^T k = 0 \qquad \text{or} \qquad v = P^{-1} A^T k \tag{30}$$

The differentiation for $\Delta_1, \Delta_2, \ldots \Delta_u$ gives:

$$B^T k = 0 \tag{31}$$

Substituting formulas (30) into (27) and using formulas (31), we obtain the (r + u) normal equations:

$$\begin{aligned} A P^{-1} A^T k + B \Delta - L &= 0 \\ B^T k &= 0 \end{aligned} \tag{32}$$

Because the square matrix $A P^{-1} A^T$ is non-singular in our problem, we may solve for k in terms of Δ.

$$k = - (A P^{-1} A^T)^{-1} (B \Delta - L) \tag{33}$$

Substituting (33) into (31) we obtain the final system of normal equations for the parameter corrections Δ.

$$\left[B^T(AP^{-1}A^T)^{-1}B\right]\Delta - B^T(AP^{-1}A^T)^{-1}L = 0 \qquad (34)$$

The mean error of an observation of unit weight is

$$m = \left(\frac{v^TPv}{r-u}\right)^{1/2} \qquad (35)$$

The feasibility of this solution depends upon the fact that the r x r matrix, $AP^{-1}A^T$, is essentially diagonal.

If the A matrix can be partitioned into diagonally arranged A_i sub-matrices, as indicated for our problem with formulas (26), it can be shown that the equation (34) may be written as

$$\sum_{i=1}^{5}\left[B^T(AP^{-1}A^T)^{-1}B\right]_i\Delta - \sum_{i=1}^{5}\left[B^T(AP^{-1}A^T)^{-1}L\right]_i = 0 \qquad (36)$$

where the index i refers to the elements of the i^{th} group. Thus it is obvious that for computational convenience the final set of normal equations may be obtained by adding separate sets of partial normal equations formed for each of the five groups. The set of partial normal equations for the i^{th} group is, according to (36):

$$\left[B^T(AP^{-1}A^T)^{-1}B\right]_i\Delta - \left[B^T(AP^{-1}A^T)^{-1}L\right]_i = 0 \qquad (37)$$

In analogy, it is possible to subpartition the matrides in (37) into units corresponding to the individual condition equations for a specific group, provided the submatrix A_i is diagonal. Then the set of partial normal equations (37) may be written as

$$\sum_{j=1}^{m}\left[B^T(AP^{-1}A^T)^{-1}B\right]_{ij}\Delta - \sum_{j=1}^{m}\left[B^T(AP^{-1}A^T)^{-1}L\right]_{ij} = 0 \qquad (38)$$

where the index j refers to the elements of the j^{th} line in the i^{th} group. Formula (38) shows that a set of partial normal equations for a specific group (formulas (37)) can be subpartitioned into as many sub-sets of partial normal equations as the corresponding A_i matrix can be subpartitioned into diagonally arranged A_{ij} submatrices. Therefore the formation of the final normal equation system can be accomplished by adding step by step the partial normal equations computed for each of the subgroups. There is an advantage in using this procedure on electronic computers, because the amount of memory space needed is almost independent of the number of points carried in the solution.

In order to conclude our investigation a study shall now be made of the characteristics of the partial normal equations for each of the smallest possible subdivisions within the five groups of observational equations as given by formulas (26). As mentioned above the matrices A_1 and A_2 are unit matrices. Therefore, for both groups for each individual conditional equation we have

$$(A_j P^{-1}_j A^T_j)^{-1} = p_j \tag{39}$$

Similarly in group 5 the A_5 matrix is a diagonally arranged sequence of non-overlapping 1x4 submatrices. Therefore, we have for each individual conditional equation in this group

$$(A_j P^{-1}_j A^T_j)^{-1} = \frac{1}{\frac{(a)^2_{1j}}{p_{1j}} + \frac{(a)^2_{2j}}{p_{2j}} + \frac{(a)^2_{3j}}{p_{3j}} + \frac{(a)^2_{4j}}{p_{4j}}} = \frac{1}{\Sigma \frac{(a)_j (a)_j}{p_j}} = \{g\}_j \tag{40}$$

Finally, we must consider groups 3 and 4 which are formed analogously. Instead of associating a set of partial normal equations with each individual observational equation as is possible in groups 1, 2 and 5, such a set must be associated with a pair of observational equations because the A_j matrices for both groups 3 and 4 are diagonally arranged sequences of non-overlapping 2x4 submatrices. For the j^{th} pair of observational equations, e.g. in group 3, we have:

$$A_{3j} = \begin{bmatrix} (a)_1 (a)_2 (a)_3 (a)_4 \\ (b)_1 (b)_2 (b)_3 (b)_4 \end{bmatrix}_{3j}$$

and consequently (41)

$$A_{3j} A^T_{3j} = \begin{bmatrix} [(a)(a)] & [(a)(b)] \\ [(a)(b)] & [(b)(b)] \end{bmatrix}_{3j}$$

Omitting the derivation and using the notation of formulas (25), the following set of partial normal equations can be formed for each pair of corresponding observational equations in groups 3 and 4, respectively.

$$\left(\left[(A)(A)(g)\right] - \left[(\dot{A})(\dot{A})(\dot{g})\right]\right)\Delta a' + \left(\left[(A)(B)(g)\right] - \left[(\dot{A})(\dot{B})(\dot{g})\right]\right)\Delta a' + \cdots\cdots \left(\left[(A)(\Delta)(g)\right] - \left[(\dot{A})(\dot{\Delta})(\dot{g})\right]\right) = 0$$

$$\left(\left[(B)(B)(g)\right] - \left[(\dot{B})(\dot{B})(\dot{g})\right]\right)\Delta a' + \cdots\cdots \left(\left[(B)(\Delta)(g)\right] - \left[(\dot{B})(\dot{\Delta})(\dot{g})\right]\right) = 0 \qquad (42)$$

$$\cdot \qquad \cdot \qquad \cdot$$

$$\left(\left[(\Delta)(\Delta)(g)\right] - \left[(\dot{\Delta})(\dot{\Delta})(\dot{g})\right]\right) = 0$$

The "." symbols indicate the cross products within each pair of observational equations belonging to a specific control point.

Therefore in formulas (42)

$$\left[(A)(A)(g)\right] = \left((A)_1(A)_1(g)_1 + (A)_2(A)_2(g)_2\right)$$

$$\left[(A)(B)(g)\right] = \left((A)_1(B)_1(g)_1 + (A)_2(B)_2(g)_2\right) \quad \text{etc.} \qquad (43)$$

and

$$\left[(\dot{A})(\dot{A})(\dot{g})\right] = \left((A)_1(A)_2 + (A)_2(A)_1\right)(g)_{1,2}$$

$$\left[(\dot{A})(\dot{B})(\dot{g})\right] = \left((A)_1(B)_2 + (A)_2(B)_1\right)(g)_{1,2} \quad \text{etc.} \qquad (44)$$

The weighting factors $(g)_1$, $(g)_2$ and $(g)_{1,2}$ are

$$(g)_1 = \frac{\Sigma\left(\frac{(b)(b)}{p}\right)_j}{N_j} \quad \text{and analogously} \quad (g)_X = \frac{\Sigma\left(\frac{(a)_Y(a)_Y}{p}\right)_j}{N_j}$$

$$(g)_2 = \frac{\Sigma\left(\frac{(a)(a)}{p}\right)_j}{N_j} \qquad (g)_Y = \frac{\Sigma\left(\frac{(a)_X(a)_X}{p}\right)_j}{N_j} \tag{45}$$

$$(g)_{1,2} = \frac{\Sigma\left(\frac{(a)(b)}{p}\right)_j}{N_j} \qquad (g)_{X,Y} = \frac{\Sigma\left(\frac{(a)_X(a)_Y}{p}\right)_j}{N_j}$$

where $N_j = \Sigma\left(\frac{(a)(a)}{p}\right)_j \cdot \Sigma\left(\frac{(b)(b)}{p}\right)_j - \left\{\Sigma\left(\frac{(a)(b)}{p}\right)_j\right\}^2$

or

$$N_j = \Sigma\left(\frac{(a)_X(a)_X}{p}\right)_j \cdot \Sigma\left(\frac{(a)_Y(a)_Y}{p}\right)_j - \left\{\Sigma\left(\frac{(a)_X(a)_Y}{p}\right)_j\right\}^2$$

From (42) it is obvious that this set of partial normal equations may be further partitioned by forming sets of partial normal equations which incorporate only the first or second terms of each of the coefficients.

For an analytical treatment the economy of the least squares solution is an important factor. Therefore the information so far obtained is summarized and the least squares solution for the process of orientation is arranged in the following way. By introducing pseudo-residuals λ we may denote in

group		
1 and 2	$v = +\lambda$	
3 and 4	$Av = -(\lambda)$	(46)
5	$A\mathring{v} = -\{\lambda\}$	

The observational equations (26) may now be written as follows

$$
\begin{array}{c}
\text{group} \\
1 \\ 2 \\ 3 \\ 4 \\ 5
\end{array}
\quad
\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ (\lambda)_3 \\ (\lambda)_4 \\ \{\lambda\}_5 \end{bmatrix}
=
\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}
\Delta -
\begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{bmatrix}
\quad
\begin{array}{c}
\text{weighting factors} \\
P \\ P \\ (g) \text{ and } (\dot{g}) \\ (g) \text{ and } (\dot{g}) \\ \{g\}
\end{array}
\qquad (47)
$$

In terms of formulas (25) the observational equations are shown on the next page.

The pseudo-residuals $((\lambda)$, and $\{\lambda\}$) are linear combinations of the original residuals v (formulas (46)). They have been introduced solely for the purpose of simplifying the computational procedure. The system (48) resembles a set of observational equations for independent indirect measurements. In the last column the corresponding weighting factors are recorded. As the first step in the process of forming the normal equations a set of partial normal equations is computed for all r observational equations according to

$$\sum_{i=1}^{r} (B^T P^* B)_i \quad - \sum_{i=1}^{r} (B^T P^* L)_i = 0 \qquad (49)$$

P^* denotes the diagonally arranged weighting factors, where $(g)_1$, $(g)_2$, $(g)_I$, $(g)_Y$ and $\{g\}$ are formed by formulas (40) and (45). The index i denotes the elements of the i^{th} line.

Now it is necessary only to take into account the fact that in groups 3 and 4 the pseudo-residuals (λ) by pairs are linear combinations of the original residuals. Denoting the number of pairs of such equations included in the r observational equations by s, another set of partial normal equations may be formed according to

$$- \sum_{k=1}^{s} (B^T_{ij} P^*_{ij} B_{ji})_k \Delta + \sum_{k=1}^{s} (B^T_{ji} P^*_{ji} L_{ij})_k = 0 \qquad (50)$$

where $P^*_{ij} = P^*_{ji} = (g)_{ij}$ denotes the cross weighting factor for a pair of corresponding observational equations. The submatrices B_{ij} and B_{ji} are the matrices of the coefficients of the unknown parameters in the pair of observa-

Group	Type of Control Point		Residuals	$\Delta\alpha'$	$\Delta\omega'$	$\Delta\kappa'$	$\Delta X_o'$	$\Delta Y_o'$	$\Delta Z_o'$	$\Delta c'$	$\Delta x_p'$	$\Delta y_p'$	$\Delta\alpha''$	$\Delta\omega''$	$\Delta\kappa''$	$\Delta X_o''$	$\Delta Y_o''$	$\Delta Z_o''$	$\Delta c''$	$\Delta x_p''$	$\Delta y_p''$	Absolute Terms $-L$	Weighting Factors
				Unknowns (Δ's)																			
1	Absolute given by X,Y,Z	for the left slot	$\lambda_x' =$	$+A_x'$	$+B_x'$	$+C_x'$	$+D_x'$	$+E_x'$	$+F_x'$	$+G_x'$	$+1$	·	·	·	·	·	·	·	·	·	·	$-\Delta l_x'$	p_x'
			$\lambda_y' =$	$+A_y'$	$+B_y'$	$+C_y'$	$+D_y'$	$+E_y'$	$+F_y'$	$+G_y'$	·	$+1$	·	·	·	·	·	·	·	·	·	$-\Delta l_y'$	p_y'
2		for the right slot	$\lambda_x'' =$	·	·	·	·	·	·	·	·	·	$+A_x''$	$+B_x''$	$+C_x''$	$+D_x''$	$+E_x''$	$+F_x''$	$-G_x''$	$+1$	·	$-\Delta l_x''$	p_x''
			$\lambda_y'' =$	·	·	·	·	·	·	·	·	·	$+A_y''$	$+B_y''$	$+C_y''$	$+D_y''$	$+E_y''$	$+F_y''$	$-G_y''$	·	$+1$	$-\Delta l_y''$	p_y''
3	Partially Absolute	given by Z	$(\lambda)_1 =$	$+(A)_1$	$+(B)_1$	$+(C)_1$	$+(D)$	·	$+(F)_1$	$+(G)_1$	$-(a)_1$	$-(a)_2$	$+(J)_1$	$+(K)$	$+(L)_1$	$-(D)$	·	$+(O)_1$	$+(P)_1$	$-(a)_3$	$-(a)_4$	$-(\Delta)_1$	$(g)_1$ $(g)_p$
			$(\lambda)_2 =$	$+(A)_2$	$+(B)_2$	$+(C)_2$	·	$+(D)$	$+(F)_2$	$+(G)_2$	$-(b)_1$	$-(b)_2$	$+(J)_2$	$+(K)_2$	$+(L)_2$	·	$-(D)$	$+(O)_2$	$+(P)_2$	$-(b)_3$	$-(b)_4$	$-(\Delta)_2$	$(g)_2$
4		given by X & Y	$(\lambda)_X =$	$+(A)_X$	$+(B)_X$	$+(C)_X$	$+(D)_X$	·	$+(F)_X$	$+(G)_X$	$-(a)_{X_1}$	$-(a)_{X_2}$	$+(J)_X$	$+(K)_X$	$+(L)_X$	$+(M)_X$	·	$-(F)_X$	$+(P)_X$	$-(a)_{X_3}$	$-(a)_{X_4}$	$-(\Delta)_X$	$(g)_X$ $(g)_{XY}$
			$(\lambda)_Y =$	$+(A)_Y$	$+(B)_Y$	$+(C)_Y$	·	$+(E)_Y$	$+(F)_Y$	$+(G)_Y$	$-(a)_{Y_1}$	$-(a)_{Y_2}$	$+(J)_Y$	$+(K)_Y$	$+(L)_Y$	·	$+(N)_Y$	$-(F)_Y$	$+(P)_Y$	$-(a)_{Y_3}$	$-(a)_{Y_4}$	$-(\Delta)_Y$	$(g)_Y$
5	Relative		$\{\lambda\} =$	$+\{A\}$	$+\{B\}$	$+\{C\}$	$+\{D\}$	$+\{E\}$	$+\{F\}$	$+\{G\}$	$-\{a\}_1$	$-\{a\}_2$	$+\{J\}$	$+\{K\}$	$+\{L\}$	$-\{D\}$	$-\{E\}$	$-\{F\}$	$+\{P\}$	$-\{a\}_3$	$-\{a\}_4$	$-\{\Delta\}$	$\{g\}$

formula (48)

tional equations under consideration. The two rows of these matrices are arranged according to the ij or ji sequence of the lines i and j, respectively.

The conformity of the two sets of partial normal equations (49) and (50) contributes markedly to the economy of the numerical solution. The final system of normal equations is the algebraic sum of the two sets of partial normal equations and may be symbolized by

$$B^T P B \Delta - B^T P L = 0 \tag{51}$$

By introducing

$$B^T P B = N \tag{52}$$

we obtain

$$\Delta = N^{-1} B^T P L \tag{53}$$

N^{-1} is the matrix of the weight coefficients of the parameters. The mean error of an observation of unit weight is computed from (35), where the numerator is obtained during the reduction of the normal equations. The modernized Gaussian algorithm[11] is suggested for this procedure. The individual v's for the absolute control points are then computed directly from the first two groups of the observational equations (48) by formulas (46) as

$$v = \lambda \tag{54}$$

From the third, fourth and fifth groups of equations (48), the (λ) and $\{\lambda\}$ values are obtained. The individual v's are then computed by formulas (30), (33) and (47). Thus we obtain for a partial control point, in the case where Z is given:

$$v'_x = \frac{(a)_1 k_1 + (b)_1 k_2}{p'_x}$$

$$v'_y = \frac{(a)_2 k_1 + (b)_2 k_2}{p'_y}$$

$$v''_x = \frac{(a)_3 k_1 + (b)_3 k_2}{p''_x} \qquad \text{where } k_1 = -(\lambda)_1(g)_1 + (\lambda)_2 g_{1,2} \tag{55}$$

$$v''_y = \frac{(a)_4 k_1 + (b)_4 k_2}{p''_y} \qquad k_2 = -(\lambda)_2(g)_2 + (\lambda)_1 g_{1,2}$$

and in the case where X and Y are given:

$$v'_x = \frac{(a_X)_1 k_1 + (a_Y)_1 k_2}{p'_x}$$

$$v'_y = \frac{(a_X)_2 k_1 + (a_Y)_2 k_2}{p'_y}$$

$$v''_x = \frac{(a_X)_3 k_1 + (a_Y)_3 k_2}{p''_x} \qquad \text{where } k_1 = -(\lambda)_X(g)_X + (\lambda)_Y(g)_{XY} \tag{56}$$

$$v''_y = \frac{(a_X)_4 k_1 + (a_Y)_4 k_2}{p''_y} \qquad k_2 = -(\lambda)_Y(g)_Y + (\lambda)_X(g)_{XY}$$

11 cf. [11]

Finally, for relative control points we obtain:

$$v'_x = \frac{\{a\}_1 k}{p'_x} \quad v'_y = \frac{\{a\}_2 k}{p'_y} \quad v''_x = \frac{\{a\}_3 k}{p'_x} \quad v''_y = \frac{\{a\}_4 k}{p'_y} \qquad \text{where } k = -\{\lambda\} \cdot \{g\} \tag{57}$$

During the computations the usual checks are made:

$$[Apv] = 0, \quad [Bpv] = 0, \text{ etc.} \quad (58) \qquad \text{and } v^T P v = [LL \cdot u] \quad (59)$$

where $[LL \cdot u]$ is obtained during the process of reducing the normal equations.

The final check is obtained by using the final orientation elements and formulas (14), (18), or (19) and (20), where either the adjusted observations $(l + v)$ and $(l' + v')$ must check (formulas (14)) or where, by introducing the corresponding adjusted observations together with the final values for the unknowns, the corresponding conditional equations must be satisfied (formulas (18), (19), (20)).

The mean errors of the unknown parameters are computed by multiplying the mean error of unit weight, which can be obtained from formula (35), by the square root of the corresponding weight coefficient which is obtained from the matrix of the weight coefficients N^{-1}.[12]

If formulas (14^a), (18^a), (19^a) and (20) are used as the basis for a least squares treatment, the arrangement of the zero-elements in the matrix of the coefficients of the observational equations is changed. The corresponding system of observational equations is shown on the following page. (formulas (48^a)).

The increase in zero elements in the third, fourth and fifth groups of the observational equations in formulas (48^a) in comparison to (48) is desirable since the presence of weighting factors causes the computation of the coefficients of the normal equations in these three groups to be more complicated than that for the first two groups. Moreover, the relative control points are usually more numerous than the absolute control points. Therefore, the system (48^a) provides a more economical solution. The main advantage of formulas (48^a), however, is that any independently obtained base line component may be introduced in the computations simply by eliminating the corresponding Δb from the least squares computation.

2) The derivation of the coefficients of the observational equations

The setting up of the observational equations (48) or (48^a) requires computing the coefficients of the matrices of the unknown parameter corrections and of the residuals. In addition, the absolute terms and the weights must

12 cf. Chapter C p. 38

Group	Type of Control Point		Residuals		$\Delta\varphi'$	$\Delta\omega'$	$\Delta\kappa'$	$\Delta X_0'$	$\Delta Y_0'$	$\Delta Z_0'$	$\Delta c'$	$\Delta x_p'$	$\Delta y_p'$	$\Delta\varphi''$	$\Delta\omega''$	$\Delta\kappa''$	Δb_x	Δb_y	Δb_z	$\Delta c''$	$\Delta x_p''$	$\Delta y_p''$	Absolute Terms $-L$	Weighting Factors
					Unknowns (Δ's)																			
1	Absolute (given by X,Y,Z)	for the left stat.	λ_x'	=	$+A_x'$	$+B_x'$	$+C_x'$	$+D_x'$	$+E_x'$	$+F_x'$	$+G_x'$	$+1$	·	·	·	·	·	·	·	·	·	·	$-\Delta l_x'$	p_x'
			λ_y'	=	$+A_y'$	$+B_y'$	$+C_y'$	$+D_y'$	$+E_y'$	$+F_y'$	$+G_y'$	·	$+1$	·	·	·	·	·	·	·	·	·	$-\Delta l_y'$	p_y'
2		for the right stat.	λ_x''	=	·	·	·	$+D_x''$	$+E_x''$	$+F_x''$	·	·	·	$+A_x''$	$+B_x''$	$+C_x''$	$+D_x''$	$+E_x''$	$+F_x''$	$+G_x''$	$+1$	·	$-\Delta l_x''$	p_x''
			λ_y''	=	·	·	·	$+D_y''$	$+E_y''$	$+F_y''$	·	·	·	$+A_y''$	$+B_y''$	$+C_y''$	$+D_y''$	$+E_y''$	$+F_y''$	$+G_y''$	·	$+1$	$-\Delta l_y''$	p_y''
3	Partially Absolute	given by Z	$(\lambda)_1$	=	$+(A)_1$	$+(B)_1$	$+(C)_1$	·	·	$+(F)_1$	$+(G)_1$	$-(a)_1$	$-(a)_2$	$+(J)_1$	$+(K)_1$	$+(L)_1$	$+(M)_1$	·	$+(O)_1$	$+(P)_1$	$-(a)_3$	$-(a)_4$	$-(\Delta)_1$	$(g)_1$, $(g)_{1,2}$
			$(\lambda)_2$	=	$+(A)_2$	$+(B)_2$	$+(C)_2$	·	·	$+(F)_2$	$+(G)_2$	$-(b)_1$	$-(b)_2$	$+(J)_2$	$+(K)_2$	$+(L)_2$	·	$+(M)_2$	$+(O)_2$	$+(P)_2$	$-(b)_3$	$-(b)_4$	$-(\Delta)_2$	$(g)_2$
4		given by X & Y	$(\lambda)_X$	=	$+(A)_X$	$+(B)_X$	$+(C)_X$	$+(D)_X$	·	·	$+(G)_X$	$-(a)_{X_1}$	$-(a)_{X_2}$	$+(J)_X$	$+(K)_X$	$+(L)_X$	$+(M)_X$	·	$+(O)_X$	$+(P)_X$	$-(a)_{X_3}$	$-(a)_{X_4}$	$-(\Delta)_X$	$(g)_X$, $(g)_{XY}$
			$(\lambda)_Y$	=	$+(A)_Y$	$+(B)_Y$	$+(C)_Y$	·	$+(E)_Y$	·	$+(G)_Y$	$-(a)_{Y_1}$	$-(a)_{Y_2}$	$+(J)_Y$	$+(K)_Y$	$+(L)_Y$	·	$+(N)_Y$	$+(O)_Y$	$+(P)_Y$	$-(a)_{Y_3}$	$-(a)_{Y_4}$	$-(\Delta)_Y$	$(g)_Y$
5	Relative		$\{\lambda\}$	=	$+\{A\}$	$+\{B\}$	$+\{C\}$	·	·	·	$+\{G\}$	$-\{a\}_1$	$-\{a\}_2$	$+\{J\}$	$+\{K\}$	$+\{L\}$	$+\{M\}$	$+\{N\}$	$+\{O\}$	$+\{P\}$	$-\{a\}_3$	$-\{a\}_4$	$-\{\Delta\}$	$\{g\}$

formula (48^a)

be computed. All coefficients are obtained by partial differentiation of the formulas (14), (18), (19) and (20) or (14ᵃ), (18ᵃ), (19ᵃ) and (20), respectively. The linearization procedure is accomplished by applying the Taylor series and neglecting second and higher order terms. Therefore, an iterative procedure must be provided for in the computation whereby the results of each cycle are introduced as approximate values in the following cycle. The iteration is repeated until the solution has converged to a pre-established accuracy level. The economy of a least squares solution is decidedly affected by the effort which goes into the computation of the coefficients of the system of observational equations. Therefore, special effort has been made to obtain these coefficients in simple combinations of auxiliary parameters.

The introduction of various combinations of the rotational components (essentially direction cosines) as auxiliaries (as given in formula (9)) and the combination of these auxiliaries with the plate coordinates as the linear auxiliary parameters u, v and w (given in formulas (11)) render the partial differential quotients in terms attractive for an analytical treatment. The partial differential quotients of these auxiliaries are:

	∂A_1	∂B_1	∂C_1	∂D_1	∂E_1	∂F_1	∂G_1	∂H_1	∂A_2	∂B_2	∂C_2	∂D_2	∂E_2	∂F_2	∂G_2	∂H_2
$\partial \alpha$	$+C_1$	0	$-A_1$	F_1	0	$-D_1$	G_2	0	C_2	0	$-A_2$	F_2	0	$-D_2$	$-G_1$	0
$\partial \omega$	D_1H_1	E_1H_1	F_1H_1	$-D_2$	E_2	$-F_2$	0	0	$-D_1H_2$	$-E_1H_2$	$-F_1H_2$	D_1	$-E_1$	F_1	0	0
$\partial \kappa$	$-A_2$	E_2H_2	$-C_2$	0	0	0	0	H_2	A_1	$-E_2H_1$	C_1	0	0	0	0	$-H_1$

	∂u	∂v	∂w
$\partial \alpha$	$+w$	0	$-u$
$\partial \omega$	$-vG_1 = u_\omega$	$-\bar{y}E_1 + cE_2 = v_\omega$	$-vG_2 = w_\omega$
$\partial \kappa$	$-(l_x - x_p)A_2 + (l_y - y_p)A_1 = -(\bar{y}G_2 + \bar{x}D_2) = u_\kappa$	$+\bar{x}E_2 = v_\kappa$	$-(l_x - x_p)C_2 + (l_y - y_p)C_1 = (\bar{y}G_1 - \bar{x}F_2) = w_\kappa$
∂c	$+D_1$	$+E_1$	$+F_1$
∂x_p	$-A_1$	$-B_1$	$-C_1$
∂y_p	$-A_2$	$-B_2$	$-C_2$

(60)

We introduce in addition the following computational auxiliaries:

$$
\begin{array}{llll}
\boxed{1} = \dfrac{x^o - x_p}{c} & \textcircled{1} = w''(Z)' & \triangle 1 = w''(X)'' + b_z u'' & \mathrm{I} = v'w'' - v''w' \\[2ex]
\boxed{2} = \dfrac{y^o - y_p}{c} & \textcircled{2} = u''(Z)'' + b_x w'' & \triangle 2 = u''(X)' & \mathrm{II} = u''w' - u'w'' \\[1ex]
\boxed{3} = A_1 - \boxed{1} \cdot D_1 & \textcircled{3} = w'(Z)'' & \triangle 3 = w'(X)' - b_x u' & \mathrm{III} = u'v'' - u''v' \\[1ex]
\boxed{4} = B_1 - \boxed{1} \cdot E_1 & \textcircled{4} = u'(Z)' - b_x w' & \triangle 4 = u'(X)'' & [1] = b_z v'' - b_y w'' \\[1ex]
\boxed{5} = C_1 - \boxed{1} \cdot F_1 & \textcircled{5} = v''(Z)'' + b_y w'' & \triangle 5 = w''(Y)' + b_z v'' & [2] = b_x w'' - b_z u'' \\[1ex]
\boxed{6} = A_2 - \boxed{2} \cdot D_1 & \textcircled{6} = v'(Z)' - b_y w' & \triangle 6 = v''(Y)' & [3] = b_y u'' - b_x v'' \\[1ex]
\boxed{7} = B_2 - \boxed{2} \cdot E_1 & & \triangle 7 = w'(Y)' - b_z v' & [4] = b_y w' - b_z v' \\[1ex]
\boxed{8} = C_2 - \boxed{2} \cdot F_1 & & \triangle 8 = v'(Y)'' & [5] = b_z u' - b_x w' \\[1ex]
\boxed{9} = \boxed{1} \cdot B_2 - \boxed{2} \cdot B_1 & & & [6] = b_x v' - b_y u'
\end{array}
\qquad (61)
$$

x^o and y^o are obtained from formulas (14) or (14^a), respectively, and $\bar{x}$ and $\bar{y}$ are computed by formulas (8). The (^{o}) symbol is used to denote approximate values.

The coefficients of the observational equations for the system of formulas (48) are now:

for formulas (14)

$$A_x = c(\boxed{1}\cdot\boxed{9} + \boxed{7})$$
$$A_y = c(\boxed{2}\cdot\boxed{9} - \boxed{4})$$
$$D_x = c\left[(1 + \boxed{1}^2)H_1 - \boxed{1}\cdot\boxed{2}\cdot H_2\right]$$
$$B_y = -c\left[(1 + \boxed{2}^2)H_2 - \boxed{1}\cdot\boxed{2}\cdot H_1\right]$$
$$C_x = -c\boxed{2}$$
$$C_y = +c\boxed{1}$$
$$D_x = -\frac{c}{q}\boxed{3}$$
$$D_y = -\frac{c}{q}\boxed{6} \qquad (62)$$
$$E_x = -\frac{c}{q}\boxed{4}$$
$$E_y = -\frac{c}{q}\boxed{7}$$
$$F_x = -\frac{c}{q}\boxed{5}$$
$$F_y = -\frac{c}{q}\boxed{8}$$
$$G_x = \boxed{1}$$
$$G_y = \boxed{2}$$
$$H_x = +1$$
$$H_y = \cdot$$
$$I_x = \cdot$$
$$I_y = +1$$

for formulas (18)

$$(A)_1 = -(1w' + 2u')$$
$$(A)_2 = -5u'$$
$$(B)_1 = -1u'_\omega + 2w'_\omega$$
$$(B)_2 = -1v'_\omega + 5w'_\omega$$
$$(C)_1 = -1u'_\chi + 2w'_\chi$$
$$(C)_2 = -1v'_\chi + 5w'_\chi$$
$$(D)_1 = -w'w''$$
$$(D)_2 = \cdot$$
$$(E)_1 = \cdot$$
$$(E)_2 = (D)_1$$
$$(F)_1 = u'w''$$
$$(F)_2 = v'w''$$
$$(G)_1 = -1D'_1 + 2F'_1$$
$$(G)_2 = -1E'_1 + 5F'_1$$
$$(H)_1 = 1A'_1 - 2C'_1 = -(a)_1$$
$$(H)_2 = 1B'_1 - 5C'_1 = -(b)_1$$
$$(I)_1 = 1A'_2 - 2C'_2 = -(a)_2$$
$$(I)_2 = 1B'_2 - 5C'_2 = -(b)_2$$

$$(J)_1 = +3w'' + 4u''$$
$$(J)_2 = 6u''$$
$$(K)_1 = 3u''_\omega - 4w''_\omega$$
$$(K)_2 = 3v''_\omega - 6w''_\omega$$
$$(L)_1 = 3u''_\chi - 4w''_\chi$$
$$(L)_2 = 3v''_\chi - 6v''_\chi$$
$$(M)_1 = -(D)_1$$
$$(M)_2 = \cdot$$
$$(N)_1 = \cdot \qquad (63)$$
$$(N)_2 = -(D)_1$$
$$(O)_1 = -u''w'$$
$$(O)_2 = -v''w'$$
$$(P)_1 = 3D''_1 - 4F''_1$$
$$(P)_2 = 3E''_1 - 6F''_1$$
$$(Q)_1 = -3A''_1 + 4C''_1 = -(a)_3$$
$$(Q)_2 = -3B''_1 + 6C''_1 = -(b)_3$$
$$(R)_1 = -3A''_2 + 4C''_2 = -(a)_4$$
$$(R)_2 = -3B''_2 + 6C''_2 = -(b)_4$$

for formulas (19)

$(A)_X = -(\triangle 1\, w' + \triangle 2\, u')$

$(A)_Y = -\triangle 6\, u'$

$(B)_X = -\triangle 1\, u'_\omega + \triangle 2\, w'_\omega$

$(B)_Y = -\triangle 5\, v'_\omega + \triangle 6\, w'_\omega$

$(C)_X = -\triangle 1\, u'_\chi + \triangle 2\, w'_\chi$

$(C)_Y = -\triangle 5\, v'_\chi + \triangle 6\, w'_\chi$

$(D)_X = -u''w'$

$(D)_Y = \cdot$

$(E)_X = \cdot$

$(E)_Y = -v''w'$

$(F)_X = u'u''$

$(F)_Y = v'v''$

$(G)_X = -\triangle 1\, D'_1 + \triangle 2\, F'_1$

$(G)_Y = -\triangle 5\, E'_1 + \triangle 6\, F'_1$

$(H)_X = \triangle 1\, A'_1 - \triangle 2\, C'_1 = -(a)_{X_1}$

$(H)_Y = \triangle 5\, B'_1 - \triangle 6\, C'_1 = -(a)_{Y_1}$

$(I)_X = \triangle 1\, A'_2 - \triangle 2\, C'_2 = -(a)_{X_2}$

$(I)_Y = \triangle 5\, B'_2 - \triangle 6\, C'_2 = -(a)_{Y_2}$

$(J)_X = \triangle 3\, w'' + \triangle 4\, u''$

$(J)_Y = \triangle 8\, u''$

$(K)_X = \triangle 3\, u''_\omega - \triangle 4\, w''_\omega$

$(K)_Y = \triangle 7\, v''_\omega - \triangle 8\, w''_\omega$

$(L)_X = \triangle 3\, u''_\chi - \triangle 4\, w''_\chi$

$(L)_Y = \triangle 7\, v''_\chi - \triangle 8\, w''_\chi$ (64)

$(M)_X = u'w''$

$(M)_Y = \cdot$

$(N)_X = \cdot$

$(N)_Y = v'w''$

$(O)_X = -(F)_X$

$(O)_Y = -(F)_Y$

$(P)_X = \triangle 3\, D''_1 - \triangle 4\, F''_1$

$(P)_Y = \triangle 7\, E''_1 - \triangle 8\, F''_1$

$(Q)_X = -\triangle 3\, A''_1 + \triangle 4\, C''_1 = -(a)_{X_3}$

$(Q)_Y = -\triangle 7\, B''_1 + \triangle 8\, C''_1 = -(a)_{Y_3}$

$(R)_X = -\triangle 3\, A''_2 + \triangle 4\, C''_2 = -(a)_{X_4}$

$(R)_Y = -\triangle 7\, B''_2 + \triangle 8\, C''_2 = -(a)_{Y_4}$

for formula (20)

$\{A\} = \boxed{1}\, w' - \boxed{3}\, u'$

$\{B\} = \boxed{1}\, u'_\omega + \boxed{2}\, v'_\omega + \boxed{3}\, w'_\omega$

$\{C\} = \boxed{1}\, u'_\chi + \boxed{2}\, v'_\chi + \boxed{3}\, w'_\chi$

$\{D\} = -I$

$\{E\} = -II$

$\{F\} = -III$ (65)

$\{G\} = \boxed{1}\, D'_1 + \boxed{2}\, E'_1 + \boxed{3}\, F'_1$

$\{H\} = -(\boxed{1}\, A'_1 + \boxed{2}\, B'_1 + \boxed{3}\, C'_1) = -\{a\}_1$

$\{I\} = -(\boxed{1}\, A'_2 + \boxed{2}\, B'_2 + \boxed{3}\, C'_2) = -\{a\}_2$

$\{J\} = \boxed{4}\, w'' - \boxed{6}\, u''$

$\{K\} = \boxed{4}\, u''_\omega + \boxed{5}\, v''_\omega + \boxed{6}\, w''_\omega$

$\{L\} = \boxed{4}\, u''_\chi + \boxed{5}\, v''_\chi + \boxed{6}\, w'_\chi$

$\{M\} = +I$

$\{N\} = +II$

$\{O\} = +III$

$\{P\} = \boxed{4}\, D''_1 + \boxed{5}\, E''_1 + \boxed{6}\, F''_1$

$\{Q\} = -(\boxed{4}\, A''_1 + \boxed{5}\, B''_1 + \boxed{6}\, C''_1) = -\{a\}_3$

$\{R\} = -(\boxed{4}\, A''_2 + \boxed{5}\, B''_2 + \boxed{6}\, C''_2) = -\{a\}_4$

In case the base line components are introduced as unknowns, the coefficients for the system of formulas (48^a) are the same for formulas (14^a) as for formulas (14) except that the arrangement has been somewhat changed. For formulas (18^a), (19^a) and (20), however, a few coefficient changes occur:

for formulas (18^a)		for formulas (19^a)		for formulas (20)	
$(D)_1 = \bullet$		$(D)_X = -II$		$\{D\} = \bullet$	
$(E)_2 = \bullet$	(66)	$(E)_Y = +I$	(67)	$\{E\} = \bullet$	(68)
$(F)_1 = -II$		$(F)_X = \bullet$		$\{F\} = \bullet$	
$(F)_2 = +I$		$(F)_Y = \bullet$			

The absolute terms are computed with

$$
\begin{aligned}
-\Delta l_x &= x^o - l_x \\
-\Delta l_y &= y^o - l_y \\
-(\Delta)_1 &= II\cdot(Z)' + [2]\, w' \\
-(\Delta)_2 &= -(I\cdot(Z)' + [1]\, w') \\
-(\Delta)_X &= II\cdot(X)' + [2]\, u' \\
-(\Delta)_Y &= -(I\cdot(Y)' + [1]\, v') \\
-\{\Delta\} &= I\cdot b_x + II\cdot b_y + III\cdot b_z
\end{aligned}
\qquad (69)
$$

The weighting factors are computed with formulas (40) and (45).

Formula (20) expresses the general case of the relative orientation. It may be of interest to consider the coefficients when conditions valid for the normal case are introduced. With $\alpha = 180^o$, $\omega = 0^o$ and $\varkappa = 0^o$, we have from formulas (9) and (11):

$$
\begin{aligned}
A_1 &= +1 & A_2 &= 0 & &\text{and} & u &= (x - x_p) \\
B_1 &= 0 & B_2 &= +1 & & & v &= (y - y_p) \\
C_1 &= 0 & C_2 &= 0 & & & w &= -c \\
D_1 &= 0 & D_2 &= 0 \\
E_1 &= 0 & E_2 &= +1 \\
F_1 &= -1 & F_2 &= 0 \\
G_1 &= 0 & G_2 &= -1 \; . \\
H_1 &= 0 & H_2 &= +1
\end{aligned}
$$

and from formulas (60) and (8):

$$u_\omega = 0 \qquad u_\chi = \bar{y} \qquad \bar{x} = -(x - x_p)$$
$$v_\omega = c \qquad v_\chi = \bar{x} \qquad \bar{y} = +(y - y_p)$$
$$w_\omega = v \qquad w_\chi = 0$$

The auxiliaries given with formulas (61) are, with $b_y = b_z = 0$, $x_p = y_p = 0$, $c' = c''$ and b_x = unity:

$$\begin{array}{llllll} \text{I} & = c(y'' - y') & [1] & = 0 & [4] & = 0 \\ \text{II} & = -c(x'' - x') & [2] & = -c & [5] & = +c \\ \text{III} & = (x'y'' - x''y') & [3] & = -y'' & [6] & = y' \end{array}$$

Thus, the coefficients (given in formulas (65)) necessary to form the corresponding observational equation are:

$$\begin{array}{llll} \{A\} & = y''x' & \{J\} & = -y'x'' \\ \{B\} & = -(c^2 + y'y'') & \{K\} & = (c^2 + y'y'') = -\{B\} \\ \{C\} & = cx' & \{L\} & = -cx'' \\ \{H\} & = -\{a\}_1 = 0 & \{Q\} & = -\{a\}_3 = 0 \\ \{I\} & = -\{a\}_2 = +c & \{R\} & = -\{a\}_4 = -c \end{array}$$

and the corresponding observational equation is, using the last line of formulas (25) and (48^a), and formulas (69):

$$\begin{aligned} c(v''_y - v'_y) = & -y''x' \, \Delta\alpha' + (c^2 + y'y'')\,\Delta\omega' - cx' \, \Delta\chi' \\ & + y'x'' \, \Delta\alpha'' - (c^2 + y'y'')\,\Delta\omega'' + cx'' \, \Delta\chi'' - c(y'' - y') \end{aligned} \tag{70}$$

Introducing $y'' - y' = p_y$ and $v''_y - v'_y = v_p$, we have since $y' \sim y''$

$$v_p = \frac{+y''x''}{c}\,\Delta\alpha'' - \frac{y'x'}{c}\,\Delta\alpha' + x''\,\Delta\chi'' - x'\,\Delta\chi' - c\left(1 + \frac{y''^2}{c^2}\right)(\Delta\omega'' - \Delta\omega') - p_y \tag{71}$$

This equation is identical e.g. with R. Finsterwalder's formula (13)[13] for the observational equation in approximately normal photography.

13 cf.[4] pp. 154, 155.

B. The process of triangulation

1) The triangulation as part of the process of orientation.

Like any other triangulation procedure, the stereophotogrammetric measuring method is mainly a means of determining the coordinates of certain target points by intersecting corresponding rays. As distinguished from intersection photogrammetry, stereophotogrammetry does not necessarily separate the orientation procedure from the triangulation phase. In fact during the simultaneous orientation of two camera positions a sufficient number of the target points to be determined are included as relative control points in the process of orientation in order to improve the over-all accuracy. The inclusion is based on the fact that any two rays which created the images on the respective plates originated from a common point and, consequently, must intersect again after the orientation is restored. Thus, by increasing the angular field of the lens the internal accuracy of the orientation is increased and, equally important, the presence and physical nature of systematic errors may be determined. The residuals of the plate measurements are obtained during the least squares adjustment for all relative control points which were included in the orientation procedure and, thus, the adjusted observations $x = l_x + v_x$ and $y = l_y + v_y$ are available for both stations. During the final check of the least squares adjustment for the orientation procedure, the corresponding u, v, w values for both stations must be computed in order to check the conditional equation (20). These values can be used in order to obtain the corresponding X,Y,Z coordinates directly from formulas (15) and (16). By means of the auxiliaries from (61) we have:

$$Z = Z'_o - \frac{[1]}{I} w' = Z'_o - \frac{[2]}{II} w' = Z'_o - \frac{[3]}{III} w' \tag{72}$$

or

$$Z = Z''_o + \frac{[4]}{I} w'' = Z''_o + \frac{[5]}{II} w'' = Z''_o + \frac{[6]}{III} w''$$

From formulas (72) and formulas (2) we obtain

$$\mu' = -\frac{[1]}{I} = -\frac{[2]}{II} = -\frac{[3]}{III} \tag{73}$$

and

$$\mu'' = \frac{[4]}{I} = \frac{[5]}{II} = \frac{[6]}{III}$$

In order to obtain the more exact numerical answer, the computation of the scaling factors μ' and μ'' should be done with auxiliaries which do not vanish. Assuming coventional geometry, the use of the auxiliaries [2], [5], and II is therefore suggested. From formulas (2) we now have:

$$\begin{aligned} X &= X'_o + \mu' u' = X''_o + \mu'' u'' \\ Y &= Y'_o + \mu' v' = Y''_o + \mu'' v'' \\ Z &= Z'_o + \mu' w' = Z''_o + \mu'' w'' \end{aligned} \tag{74}$$

where $\mu' = -\frac{[2]}{II}$ and $\mu'' = \frac{[5]}{II}$

The computation of the mean errors of the final coordinates will be discussed in a separate chapter.

2) The triangulation as an independent computational procedure.

Occasionally, during the process of orientation, it is not desirable to carry as relative control points all the points whose coordinates are to be determined. For those points not included in the process of orientation, an independent coordinate determination is necessary. The positions of the corresponding rays are determined by the elements of orientation as obtained from a least squares adjustment and the four measured plate coordinates. It is obvious that, because of unavoidable measuring errors, these rays will not intersect. They must be made to intersect so that the sum of the squares of the corrections to be applied to the original plate measurements is a minimum. Such a least squares adjustment of conditioned observations may be based on formulas (20). Since there is only one conditional equation present, we have only one normal equation which is used to determine the correlate k. Then the residuals are easily computed according to formulas (57). The final check is obtained by introducing the adjusted observations into the original conditional equation (20). From here the procedure of computing the final X,Y,Z coordinates by formulas (74) follows the steps outlined in the preceding paragraph.

The conditional equation for the triangulation adjustment is, using formulas (25), (61), (65) and (69):

$$\{a\}_1 v'_x + \{a\}_2 v'_y + \{a\}_3 v''_x + \{a\}_4 v''_y - \{\Delta\} = 0 \tag{75}$$

and the corresponding normal equation is

$$Nk = \{\Delta\} \tag{76}$$

or

$$k = \frac{\{\Delta\}}{N} \tag{77}$$

where

$$N = \sum \frac{\{a\}\{a\}}{p} \tag{78}$$

As in formulas (57) the resdiuals are:

$$v'_x = \frac{\{a\}_1}{p'_x} k \qquad v''_x = \frac{\{a\}_3}{p''_x} k$$
$$v'_y = \frac{\{a\}_2}{p'_y} k \qquad v''_y = \frac{\{a\}_4}{p''_y} k \tag{79}$$

We have the check:

$$v^T P v = + \{\Delta\} \cdot k \tag{80}$$

The computation of the mean errors of the final coordinates will be discussed in the next chapter.

C. The determination of the mean errors of the observations, of the elements of orientation and of the triangulation results.

The mean error of an observation of unit weight denoted by m is computed from formulas (35)

$$m = \sqrt{\frac{v^T P v}{n - u}}$$

$v^T P v$ may be obtained directly from the reduction of the normal equations according to formula (59) or by adding the squares of the individual weighted v values. Thus the mean error of an observation l before adjustment is

$m_1 = \frac{m}{\sqrt{P_1}}$. Sometimes the computations of m directly from the original measurements e.g., using the differences of multiple observations, may lead to a value of greater physical significance. The discrepancies between the m values computed by the different methods provide means to investigate the presence of systematic errors.

The inverse of the matrix of the coefficients in the final normal equation system (52) is the matrix of the weighting factors. The diagonal elements are the squares of the weighting factors of the corresponding unknown parameters. Multiplying these weighting factors by m gives the corresponding mean errors.

Often it will be of interest to know the mean error of a function of the unknown parameters.[14] We may consider the function

$$F(\alpha', \omega' \ldots y_p', \alpha'', \omega'', \ldots y_p'') = 0 \tag{81}$$

In case the function is not linear, we apply the Taylor series. This gives $F = f_o + f_1 \partial\alpha' \ldots\ldots$. The f_i values are thus the partial differential quotients of F with respect to the unknown parameters. If we introduce the vector

$$(f_1 f_2 \ldots\ldots f_{18}) = f^T \tag{82}$$

the weight of the function F denoted by P_F, together with formula (52), is

$$P_F = f^T N^{-1} f \tag{83}$$

The mean error of F denoted by m_F is thus

$$m_F = m \sqrt{\frac{1}{P_F}} = m(f^T N^{-1} f)^{-\frac{1}{2}} \tag{84}$$

By applying the above procedure, we may determine the mean error of any function of the orientation elements. For example, for b_x the function F is

$$F = X_c'' - X_c' \tag{85}$$

14 cf. [6] pp. 99 - 100

and, according to formula (81), we obtain with the sequence used in formulas (48)

$$
\begin{aligned}
f_4 &= -1 \\
f_{13} &= +1 \quad \text{All other } f_i \text{ - values are zero.}
\end{aligned}
\tag{86}
$$

The weight P_{b_x} and the mean error m_{b_x} are now computed by formulas (83) and (84), respectively. If the base line components b_x, b_y and b_z have been considered as unknown parameters (cf. formulas (48)a), the terms $\frac{1}{P_{b_x}}$, $\frac{1}{P_{b_y}}$ and $\frac{1}{P_{b_z}}$ are obtained directly from the corresponding diagonal terms of the matrix of the weighting factors.

The mean errors of the triangulated X, Y, Z coordinates are denoted by m_X, m_Y and m_Z. The procedure of computing these mean errors depends upon whether the point was included in the least squares adjustment of the orientation process as a relative control point (see B/1) or whether the point was afterwards determined by a least squares adjustment based on four independent plate measurements together with the independently computed elements of orientation (see (B/2). The latter case may be treated by analogy to the already discussed problem of determining the mean error of a function of the unknown parameters. After the adjustment by formulas (75-79) each of the spatial coordinates X, Y, Z may be expressed by formula (74) as a function F of the elements of orientation and four independently adjusted plate measurements. Denoting the partial differential quotients with respect to the elements of orientation and the plate measurements by $f_1, f_2 \ldots f_{18}$ and F_1, F_2, F_3 and F_4, respectively, we may introduce the vectors

$$
\begin{aligned}
f &= (f_1 f_2 \ldots f_{18}) \\
\text{and} \quad F &= (F_1 F_2 F_3 F_4)
\end{aligned}
\tag{87}
$$

The propagation of the errors of the elements of orientation into the function F is, with (84)

$$m_{(f)} = m \left(f^T N^{-1} f \right)^{-\frac{1}{2}} = m \cdot Q^{\frac{1}{2}}_{(f)} \tag{88}$$

The propagation of the errors of the independently adjusted plate measurements is

$$m_{(F)} = m \cdot Q^{\frac{1}{2}}_{(F)} \tag{89}$$

where $Q_{(F)}$ may be obtained in the usual way during the reduction of the corresponding normal equation. Since the orientation parameters are not correlated to the plate measurements used in the triangulation phase, the combined mean error of F is computed by the Gaussian law of propagation

$$m_F = m \left(Q_{(f)} + Q_{(F)} \right)^{\frac{1}{2}} \tag{90}$$

In approximately normal photography the influence of the elements of interior orientation on the determination of X, Y, Z can be neglected because in such cases the elements of exterior orientation compensate for the effect of the errors of the elements of interior orientation.[15] However, the accuracy of the elements of interior orientation is important in interpreting the mean errors of the elements of exterior orientation.[16]

If the point to be triangulated is included as a relative control point in the process of orientation, we must determine the mean error of a function of the unknown parameters as well as of the adjusted observations. This problem was treated first by Helmert.[17]

For a detailed presentation of this problem in matrix notation see [2].[18]

It follows that we may introduce the vector

$$\mathbf{F}^T = (f_1,\ f_2, \ldots\ldots f_{18},\ F_1,\ F_2,\ F_3,\ F_4) \tag{91}$$

where the f and F values have the same meaning as in formulas (87).

The mean error of the function F is now given by

$$m_F = m(\mathbf{F}^T \mathbf{R}\, \mathbf{F})^{\frac{1}{2}} \tag{92}$$

where $\mathbf{R}$ denotes the covariance matrix of the vector of the adjusted parameters and the adjusted observations. The numerical treatment of this problem is quite cumbersome.

The analytical method permits the use of a large number of relative and absolute control points. The matrix of the weighting coefficients N^{-1} (e.g. formula (88)) decreases as the overdetermination increases. Thus with a strong overdetermination the result obtained from formula (90) provides a satisfactory approximation for the determination of the mean errors of the triangulated coordinates, making the utilisation of formula (92) unnecessary.

V. THE APPLICATION OF THE ANALYTICAL METHOD TO THE PROBLEM OF CONTROL EXTENSION

The process of strip triangulation on the restitution instruments is based upon the technique of orienting individual photographs so that the orientation of a preceding photograph and the spatial position of at least one point of the preceding model are enforced. This procedure, as well as the separation of

15 cf. [1]

16 cf. [9]

17 cf. [5] pp. 215 - 222

18 cf. [2] page 14.

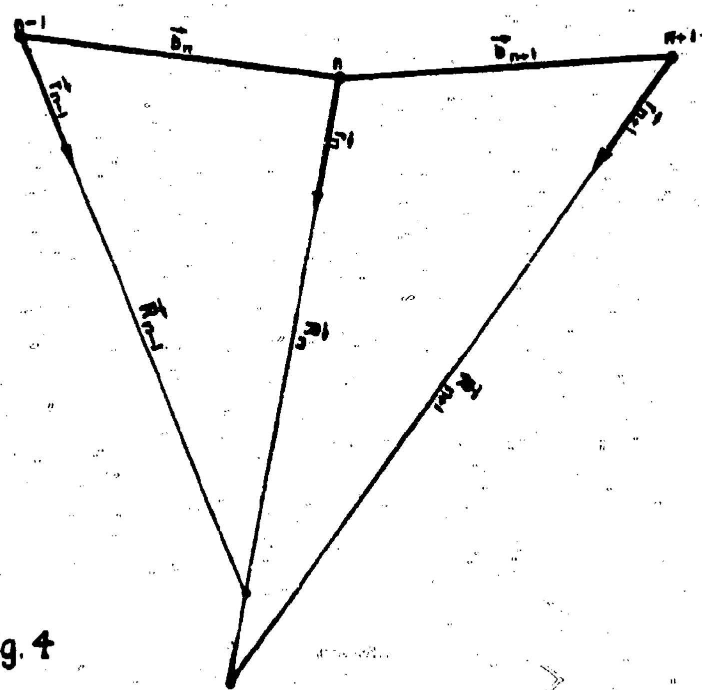


Fig. 4

relative and absolute orientation in the restitution of an individual model, is dictated by the physical properties of the restitution equipment. The enforcement of the above mentioned parameters in an aerial triangulation is comparable to a procedure which breaks up the least squares adjustment of a triangulation net into a series of adjustments of individual configurations, enforcing the result of one configuration, with respect to orientation and scale, into the following one. However, the application of such a procedure does not meet the error theoretical requirements. In order to carry the orientation and scale over a strip of overlapping photographs it is necessary to satisfy the condition that rays originating from three consecutive camera stations intersect for at least one point located in the area common to three photographs. Analytically, such a point gives rise to three independent conditional equations. If we denote three consecutive photographs by (n-1), n, (n+1), an obvious method of setting up the three conditional equations is the triple application of formula (20) which expresses the condition of intersection for pairs of rays originating from the stations (n-1) and n, n and (n+1), (n-1) and (n+1). However, this method does not lead to a result if the three rays are located in or close to a plane. This situation is common in strip triangulation. From Figure 4 we see that three independent conditional equations can be obtained by applying twice the above mentioned condition of intersection expressed by formula (20), e.g. for (n-1) and n, and n and (n+1).

The condition that these two intersections occur at a common point is equal to the condition that the vector $\vec{R}_n$ is common to the two loops $\vec{R}_{n-1} - \vec{R}_n - \vec{b}_n = 0$ and $\vec{R}_n - \vec{R}_{n+1} - \vec{b}_{n+1} = 0$. Thus, with formulas (1) we have the condition that $\mu_n \vec{r}_n$ in the first loop must equal the corresponding expression in the second loop. Because the vector $\vec{r}_n$ is identical in both loops, an independent conditional equation is obtained by equalizing the scale factors μ_n obtained from the two loops. Denoting the components of the first and second loops by the indices 1 and 2, respectively, we have:

$$\mu_1^{\prime\prime} = \mu_2^{\prime} \tag{93}$$

Formulas (73) give for each scale factor μ three quotients corresponding to the projection of the vector $\vec{R}$ into the coordinate planes. If we arrange the ground reference in such a way that the X-axis points approximately in the direction of the strip, it is obvious that the terms involved in the auxiliaries [2] , [5] and II, respectively, (formulas (61)) will always be different from zero. Therefore we choose as the third conditional equation using (93), (73) and (61)

$$[2]\ {}_2II_1 + [5]\ {}_1II_2 = 0 \tag{94}$$

The coefficients of the corresponding observational equation are obtained from formula (94) as the partial differential quotients with respect to the unknown parameters.

We introduce the following auxiliaries, which are similar to those developed in formulas (61):

$$
\begin{aligned}
&(u'u'' + w'w'') = IV \\
&(b_x u' + b_z w') = [7] \qquad && (u'w''_\omega - w'u''_\omega) = [19] \\
&(b_x u'' + b_z w'') = [8] && (u''w'_\omega - w''u'_\omega) = [20] \\
&(b_x w'_\omega + b_z u'_\omega) = [9] && (u'w''_\kappa - w'u''_\kappa) = [21] \\
&(b_x w''_\omega - b_z u''_\omega) = [10] && (u''w'_\kappa - w''u'_\kappa) = [22] \\
&(b_x w'_\kappa - b_z u'_\kappa) = [11] && (u'F''_1 - w'D''_1) = [23] \\
&(b_x w''_\kappa - b_z u''_\kappa) = [12] && (u''F'_1 - w''D'_1) = [24] \\
&(b_x F'_1 - b_z D'_1) = [13] && (u'C''_1 - w'A''_1) = [25] \\
&(b_x F''_1 - b_z D''_1) = [14] && (u''C'_1 - w''A'_1) = [26] \\
&(b_x C'_1 - b_z A'_1) = [15] && (u'C''_2 - w'A''_2) = [27] \\
&(b_x C''_1 - b_z A''_1) = [16] && (u''C'_2 - w''A'_2) = [28] \\
&(b_x C'_2 - b_z A'_2) = [17] \\
&(b_x C''_2 - b_z A''_2) = [18]
\end{aligned}
\tag{95}
$$

Thus, if arranged in the same sequence as in formulas (65), the coefficients of the corresponding observational equation are:

$$
\begin{aligned}
\{A\} &= +[7]_1 II_2 - [2]_2 IV_1 \\
\{B\} &= -[9]_1 II_2 + [2]_2 [20]_1 \\
\{C\} &= -[11]_1 II_2 + [2]_2 [22]_1 \\
\{D\} &= +w'_1 II_2 \\
\{E\} &= 0 \qquad\qquad (n-1)^{th} \text{ photograph} \\
\{F\} &= -u'_1 II_2 \\
\{G\} &= -[13]_1 II_2 + [2]_2 [24]_1 \\
-\{a\}_1 = \{H\} &= +[15]_1 II_2 - [2]_2 [26]_1 \\
-\{a\}_2 = \{I\} &= +[17]_1 II_2 - [2]_2 [28]_1
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
\{J\} &= -[5]_1\mathrm{IV}_2 + [2]_2\mathrm{IV}_1 \\
\{K\} &= +[5]_1[20]_2 - [2]_2[19]_1 \\
\{L\} &= +[5]_1[22]_2 - [2]_2[21]_1 \\
\{M\} &= -(w_2''\mathrm{II}_1 + w_1'\mathrm{II}_2) \\
\{N\} &= \quad \cdot \\
\{O\} &= +u_2''\mathrm{II}_1 + u_1'\mathrm{II}_2 \\
\{P\} &= +[5]_1[24]_2 - [2]_2[23]_1 \\
-\{a\}_3 = \{Q\} &= -[5]_1[26]_2 + [2]_2[25]_1 \\
-\{a\}_4 = \{R\} &= -[5]_1[28]_2 + [2]_2[27]_1
\end{aligned}
$$

n^{th} photograph

(96) cont.

$$
\begin{aligned}
\{S\} &= +[5]_1\mathrm{IV}_2 - [8]_2\mathrm{II}_1 \\
\{T\} &= -[5]_1[19]_2 + [10]_2\mathrm{II}_1 \\
\{U\} &= -[5]_1[21]_2 + [12]_2\mathrm{II}_1 \\
\{V\} &= +w_2''\mathrm{II}_1 \\
\{W\} &= \quad \cdot \\
\{X\} &= -u_2''\mathrm{II}_1 \\
\{Y\} &= -[5]_1[23]_2 + [14]_2\mathrm{II}_1 \\
-\{a\}_5 = \{Z\}_a &= +[5]_1[25]_2 - [16]_2\mathrm{II}_1 \\
-\{a\}_6 = \{Z\}_b &= +[5]_1[27]_2 - [18]_2\mathrm{II}_1
\end{aligned}
$$

$(n+1)^{th}$ photograph

The corresponding $\{\Delta\}^*$ value is computed from (92)

$$-\{\Delta\}^* = [2]_2\mathrm{II}_1 + [5]_1\mathrm{II}_2 \tag{97}$$

It should be noted that all the components used to form the auxiliaries given by formulas (95) are already available from the setting up of the other two observational equations based on formulas (20).

An analytical treatment of the two methods of extension shows the difference between a simulation by instruments and the least squares adjustment. Assuming a strip of approximately vertical photographs flown with 2/3 overlap, the unique solution may be considered first. The control is schematically arranged as shown in Figure 5 for n = 6 photographs.

From Figure 5 we obtain the following independent observational equations:

point no.	type of control point	number of observational equations	number of conditions of intersection satisfied	formulas describing simulation by instrument	formulas describing least squares adjustment
1	△	4	1	(14)	(14)
2	△	4	1	(14)	(14)
3	○	2	2	(20)	(20)
4	○	2	2	(20)	(20)
5	⊠	4	2	(18) & (14)	(18)
6	○	2	2	(20)	(20)
7	¤	3	2	(20) & (14)	(20) & (94)
8	○	2	2	(20)	(20)
9	○	2	2	(20)	(20)
10	¤	3	2	(20) & (14)	(20) & (94)
11	○	2	2	(20)	(20)
12	¤	3	2	(20) & (14)	(20) & (94)
13	○	1	1	(20)	(20)
14	○	1	1	(20)	(20)
15	○	1	1	(20)	(20)
$\sum$		36	25		

This tabulation shows that in both cases we have 36 observational equations for the determination of the 6n orientation parameters, n being 6 in our case. Furthermore, five pairs of rays within each pair of consecutive bundles must intersect in order to satisfy the rigorous geometry. Thus, if n denotes the number of photographs in the strip, we need $(n-1) \cdot 5$ conditions of intersection. For $n = 6$, twenty-five conditions of intersection must be satisfied as shown in the tabulation.

In Figure 6 the 36 observational equations are shown.

From Figure 5 we see that five points are common to two photographs in each of the end models and ten points are common to three photographs in the central portion of the strip. Consequently, allowing two residuals for each image point, we have $2(5 \cdot 2 + 10 \cdot 3) = 80$ residuals. In Figure 6 these residuals are represented in the corresponding **A** matrix by open circles or dots, respectively, depending upon whether the corresponding observational equation belongs to the least squares solution or to a set of equations simulating the extension

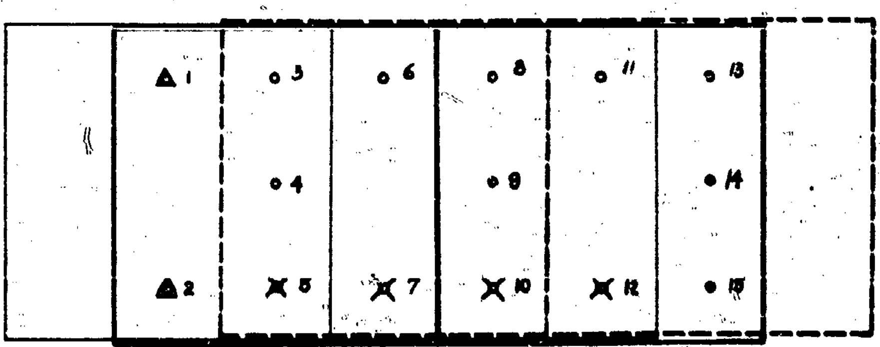
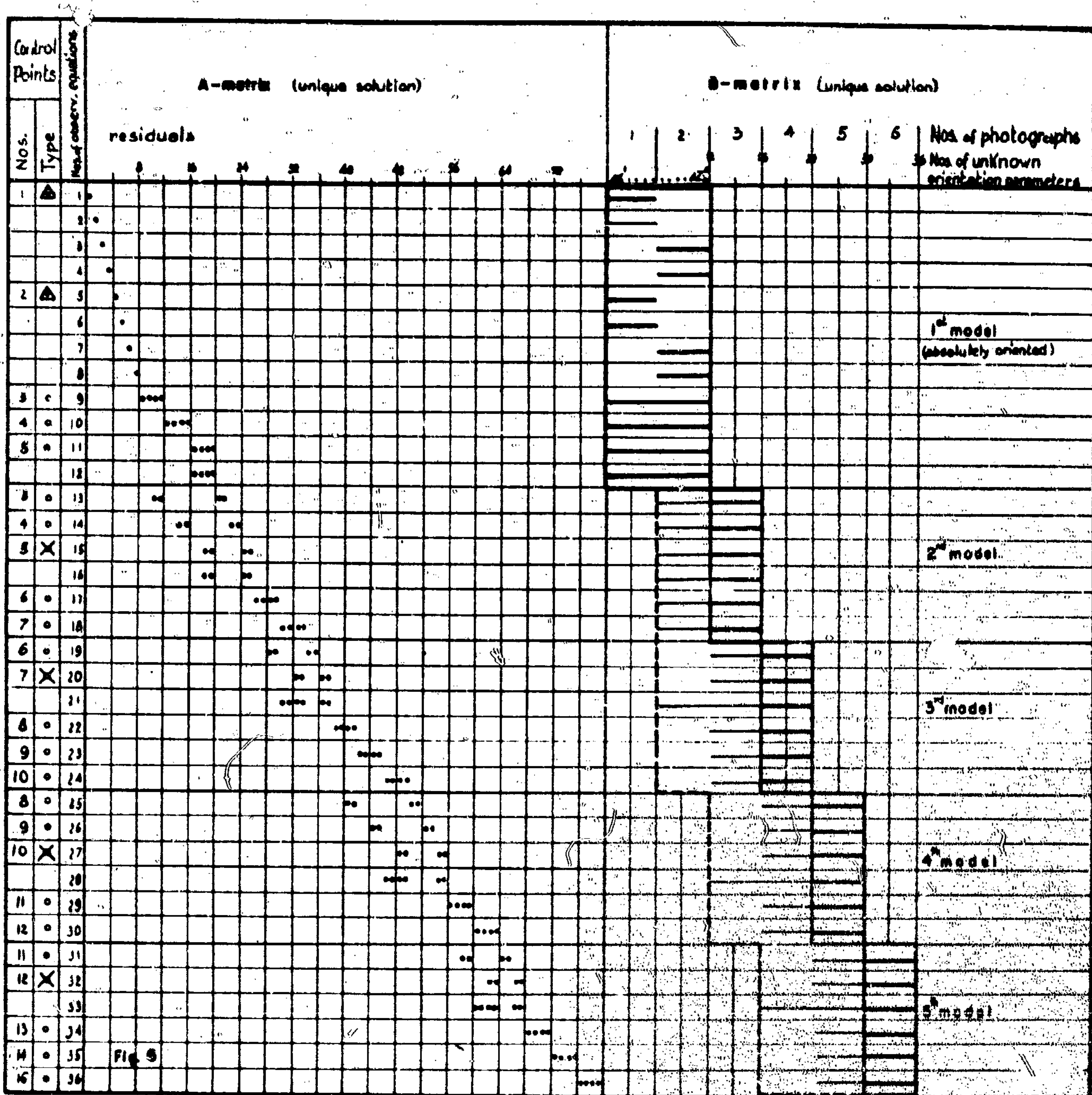


Fig. 5

- ▲ absolute control points given by XYZ
- • partially absolute points given by Z
- ○ relative points
- ✕ points used to carry over



Control Points
Nos.
Type
Nos. of observ. equations
A-matrix (unique solution)
residuals
B-matrix (unique solution)
Nos. of photographs
Nos. of unknown orientation parameters
1st model
(absolutely oriented)
2nd model
3rd model
4th model
5th model

Fig. 5

procedure as performed on restitution equipment. Correspondingly, the sequence of the coefficients of the unknown parameter corrections is given by thin and heavy lines in the **B** matrix. It is obvious that the conventional method of strip triangulation is divided into a sequence of independent extensions, thus allowing for the computation of the six unknown elements of orientation for each consecutive photograph separately (heavy lines). Correspondingly, the **A** matrix is a sequence of diagonally arranged non-overlapping row vectors (dots). However, such treatment will be correct only in the case of a unique solution where all residuals are equal to zero and, therefore, the numerical solution will not be influenced by incomplete conditional equations. A least squares adjustment, however, must be based on the rigorous conditional equations designated by circles and thin lines in Figure 6.

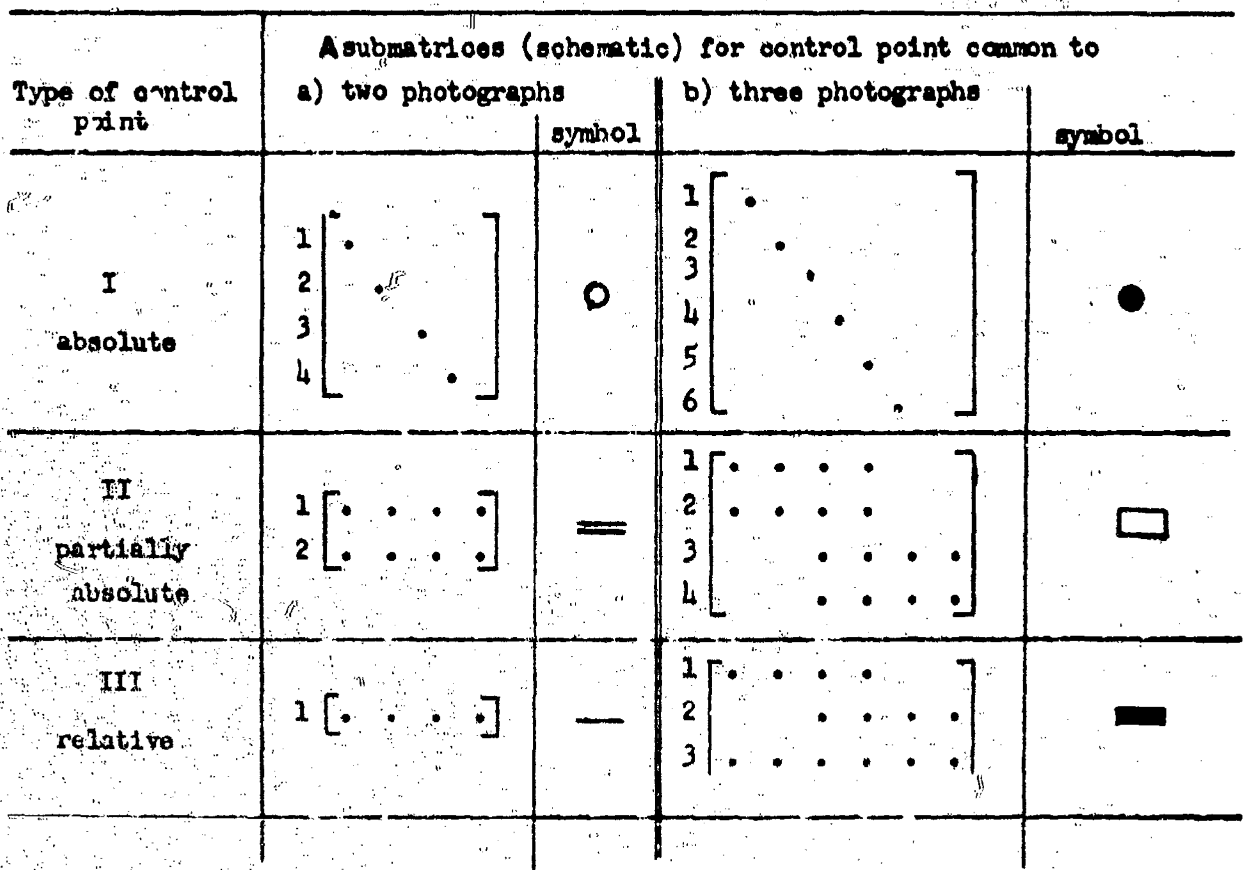


FIGURE 7

The corresponding system of observational equations can be written in analogy to formula (26) as follows:

$$\begin{bmatrix} \mathbf{A}_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \mathbf{A}_2 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \mathbf{A}_3 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \mathbf{A}_i \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \cdot \\ \cdot \\ \mathbf{v}_i \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \cdot \\ \cdot \\ \mathbf{B}_i \end{bmatrix} \mathbf{\Delta} - \begin{bmatrix} \mathbf{L}_1 \\ \mathbf{L}_2 \\ \mathbf{L}_3 \\ \cdot \\ \cdot \\ \mathbf{L}_i \end{bmatrix} = 0 \qquad (98)$$

where $\mathbf{A}_i$ are the coefficient matrices of the residual vectors $\mathbf{v}_i$ corresponding to the i control points involved. It is seen from Figure 6 and Figure 7, that six different types of $\mathbf{A}_i$ submatrices exist, depending upon what type of control point is used and whether the point is common to two or three photographs.

The $\mathbf{A}$ submatrices of the types I_a, II_a and III_a are obviously identical with the corresponding expressions in formulas (25).

It follows from formula (98) that the geometric conditions in a strip can be simulated by a set of conditional equations explicit in terms of the residual vectors $\mathbf{v}$. Any number of any type of control point situated at any portion of the strip can be handled readily. In addition any number of independently determined orientation parameters can be enforced in the solution at any time.

To show the procedure of a strip triangulation based on a least squares solution, let us consider a strip of 5 photographs with 2/3 overlap and a pass point distribution as shown in Figure 8.

Twelve points are common to two photographs and 18 points are common to three photographs. Consequently, we have $(12 \cdot 2 + 18 \cdot 3) \times 2 = 156$ residuals, which are arranged in $\mathbf{A}$ submatrices according to Figure 7.

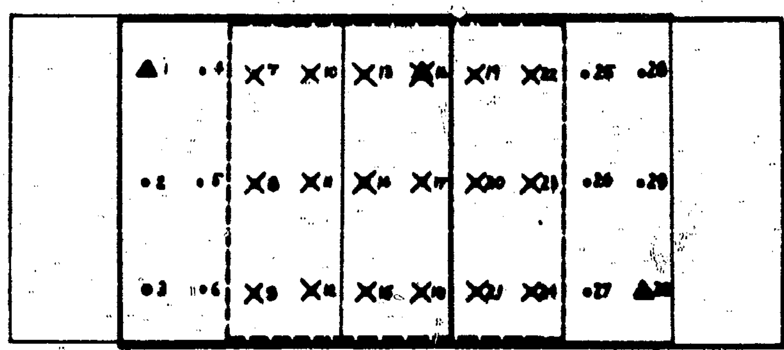


Fig. 8

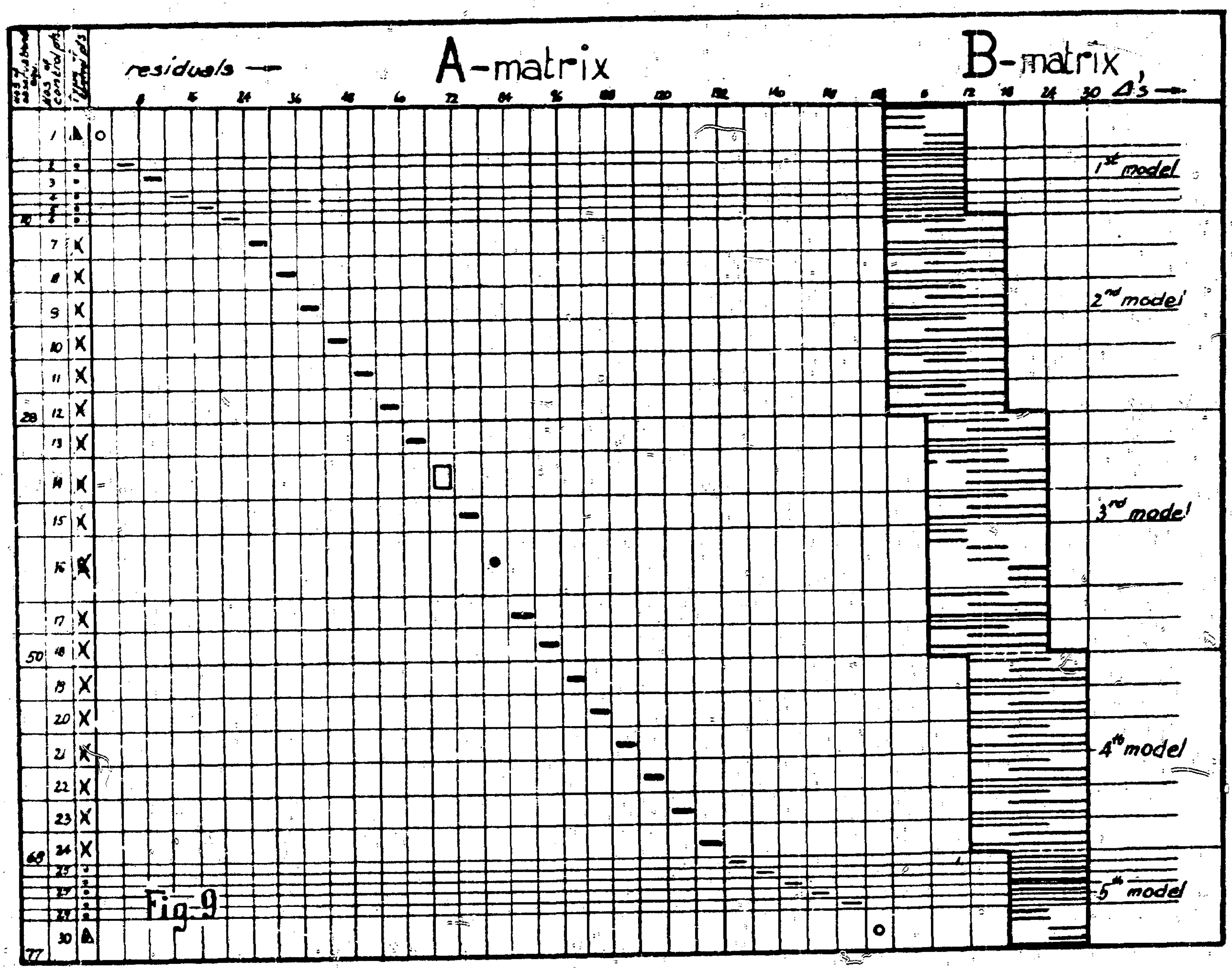

residuals
A-matrix
B-matrix
Δ's
1st model
2nd model
3rd model
4th model
5th model

Fig. 9

The number of observational equations is given by:

point no.	type	number of equations	point no.	type	number of equations
1	△	4	16	¤	6
2	○	1	17	¤	3
3	●	2	18	¤	3
4	○	1	19	¤	3
5	○	1	20	¤	3
6	○	1	21	¤	3
7	¤	3	22	¤	3
8	¤	3	23	¤	3
9	¤	3	24	¤	3
10	¤	3	25	○	1
11	¤	3	26	○	1
12	¤	3	27	○	1
13	¤	3	28	○	1
14	¤	4	29	○	1
15	¤	3	30	△	4
$\sum$					77

Thus there are 77 observational equations for $n \cdot 6 = 5 \cdot 6 = 30$ unknown parameters.

With the symbols used in Figure 7, the **A** and **B** matrices of the observational equations are schematically shown in Figure 9. The **A** matrix consists of diagonally arranged $\mathbf{A}_i$ submatrices. The final set of normal equations can be formed by adding sets of partial normal equations as outlined on pages 22 and 23. Thus the analytical treatment of a least squares solution for a strip triangulation is reduced to the problem of inverting a symmetrical $r \times r$ matrix, the coefficient matrix of the unknown orientation parameters. Since the general case deals with six unknown parameters for each photograph we have for a strip of n photographs, $r = 6n$. The feasibility of the proposed

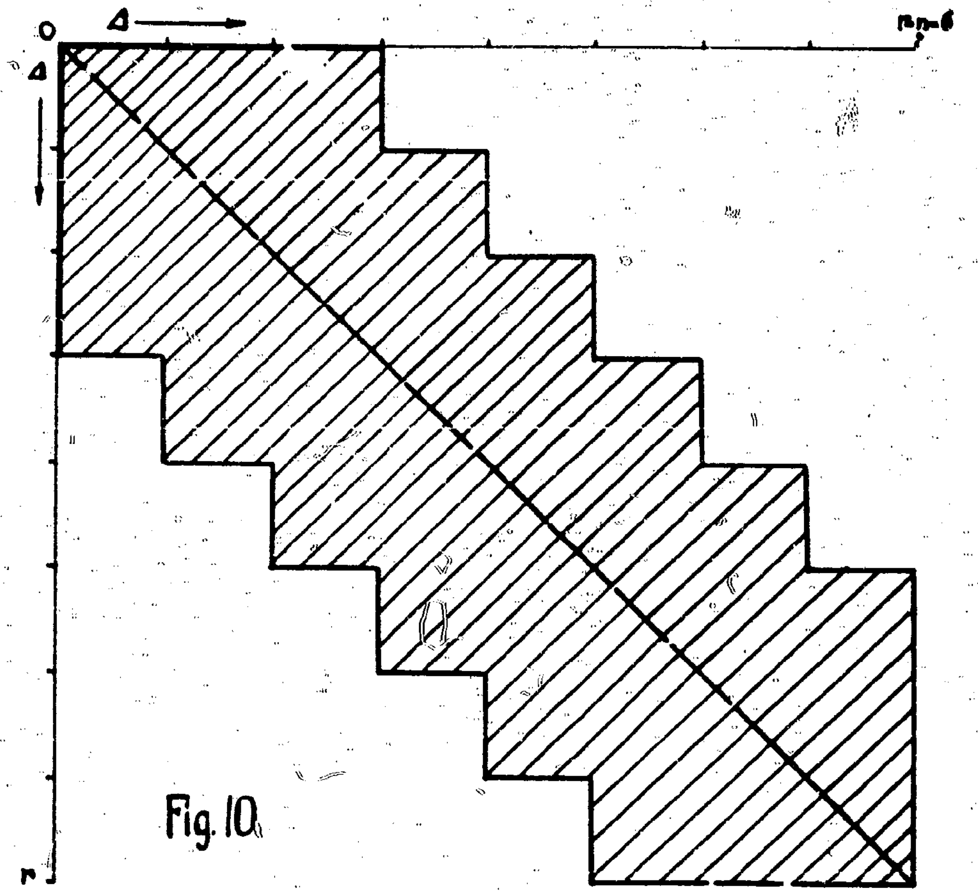


Fig. 10

A symmetrical system of $r \times r$ normal equations.
Only the shaded area is filled with coefficients. All other places are filled with zeros.

approach to the problem of aerial triangulation appears, therefore, to be based upon the possibility of inverting large matrices. This problem will be simplified because the individual **B** matrices in formula (98) include only coefficients corresponding to the unknown parameters combined in the individual vectors Δ. Thus the r x r matrix of the coefficients is only partially filled, namely, in the neighborhood of the diagonal as shown in Figure 10. This fact suggests the application of the relaxation technique of Gauss-Seidel or a similar approach. The determination of the roots of the normal equations can be based upon an iterative process which takes place as a subroutine within each cycle of iteration of the general solution.

If the pass points are selected in such a way that at least 5 relative control points are common to each section of triple overlap, those models formed by the combination of photographs taken at every other camera position are included in the computations. The favorable geometry of the thus extended base line is otherwise obtainable only by convergent photography. If we assume 2/3 overlap between the consecutive photographs (see Figure 8), the triangulation strip consists of a series of sections with triple overlap. Consequently, as a general rule, we will obtain about 3 times as many observational equations as we have relative control points. In addition to having favorable geometry, we have a redundant number of points which are used to distrubute the errors of observation and therefore increase the accuracy and reliability of the result. Photographs taken with lenses of hyper-wide angle of view (120°), which are expected to be available in the future, should provide the necessary economy in a strip with 2/3 overlap.

The basic idea underlying the solution presented for a strip triangulation can obviously be applied to the problem of a least squares solution for a block adjustment. Depending upon the degree of side lap, the pattern of the **A** submatrices will vary according to the change in the number of photographs which have in common the images of certain control points. However, the matrix of the final system of observational equations will again consist of a sequence of diagonally arranged **A** submatrices, so that the final normal equation system may be formed stepwise. It should be noted that the application of formula (20), as explained on page 42, for the purpose of combining adjoining models can now be used advantageously because in this case the rays to be combined do not lie in or close to a plane.

VI. CONCLUDING REMARKS

The feasibility of the presented analytical treatment of a photogrammetric evaluation problem depends upon the availability of electronic computing machines with large computing capacity and large storage facilities. The high speed with which such computers are able to perform a multitude of arithmetic operations suggests the practicability of iterative processes. Therefore, the least squares solution is advantageously based on equations which yield first order differential corrections to approximate values of the unknown parameters, the reduction being repeated until the solution has converged to a pre-established accuracy level. The large computing capacity makes it feasible to design the solution for the most general case, thus eliminating the need for special solutions. In addition, the overall accuracy can be increased by over-determining the solution through a redundant number of observations. Any type of given information in the form of pass points or orientation elements can be readily introduced.

Numerous hand computed examples have been carried out for the method described in this report. At present a universal code is being prepared at the Ballistic Research Laboratories for the problem of orienting individual models and triangulating the spatial coordinates of individual points on such models.

It is assumed that such a code together with an automatically recording stereocomparator will provide a solution of maximum precision and economy. It appears that such a solution is applicable to all photogrammetric problems in which either the elements of orientation or the spatial coordinates of numerous target points must be determined.

Hellmut H. Schmid.
HELLMUT H. SCHMID

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