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DEPARTMENT OF ENGINEERING

generalized operational calculus for time-varying networks

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GENERALIZED
OPERATIONAL CALCULUS
FOR TIME-VARYING
NETWORKS

Part I E. C. Ho

Part II H. Davis

[Th]is report announces some new results and observations brought about by a review of the work of Aseltine, Davis and Trautman, relative to analysis and synthesis of linear systems whose parameters vary with time.

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FOREWORD

The work described in this report, "Generalized Operational Calculus for Time-Varying Networks," was carried out under the supervision and technical responsibility of DeForest L. Trautman, and is part of a coordinated program in Network Research.

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July, 1954

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GENERALIZED OPERATIONAL CALCULUS FOR TIME-VARYING NETWORKS

Part I

Er-Chun Ho

Part I of this report concerns (a) the inverse transform for linear time-varying networks and (b) the existence and applicability of the transform method for time-varying networks.

1. Introduction

With the introduction of the transformation technique, analysis and synthesis of a certain class of linear time-varying networks may be greatly simplified. The method has the general advantages of the Laplace transformation for lumped constant networks. The original studies of Aseltine[1] on the subject covered the development of a general technique for obtaining the direct integral transformation for a given time-varying network and its applications to the analysis and synthesis of two specific time-varying networks.

From the engineering point of view, the work of Aseltine is only a beginning and suggests many areas for further studies. Among them, problems such as the method of developing inversion formula and the general considerations of the applicability of the transformation method to time-varying networks are of particular interest. This report summarizes the general results of the studies on these topics (see also ref. 2,3).

First, a brief outline of the Aseltine method of finding the direct transform for a given time-varying network will be given. Then, a general method of finding the inverse transformation integral once the direct transformation integral is known will be discussed. Finally, the existence and applicability of a transform for a given time-varying network will be considered in some detail.

II. The Direct Transform

A linear time-varying network referred to in this report is a network (or system) whose behavior is described by a second order linear differential equation with variable coefficients in the form:

$$a(t)\, q''(t) + b(t)\, q'(t) + d^2 q(t) = v(t). \tag{1}$$

Equation (1) will be called the network equation. The direct integral transformation for eq. (1) may be developed according to the method given by Aseltine[1] as follows:

The desired transformation, a direct analogy of Laplace transform, is defined as

$$T\left[q(t)\right] = Q(\eta) = \int_0^\infty q(t)\, h(\eta, t)\, dt \tag{2}$$

where $h(\eta,t)$ is the kernel of the transformation. The kernel is so chosen that eq. (1) can be converted into the following algebraic equation by the application of the transformation

$$\left[f(\eta) + d^2\right] Q(\eta) = V(\eta) + \left[\begin{matrix}\text{Initial}\\ \text{Conditions}\end{matrix}\right]. \tag{3}$$

The method of finding the suitable transform kernel is as follows:

Define the transform kernel by

$$h(\eta,t) = g(t)\, k(\eta,t). \tag{4}$$

In eq. (4), $k(\eta,t)$ is a solution of the kernel equation

$$a(t)\, k_{tt}(\eta,t) + b(t)\, k_t(\eta,t) + F(\eta)\, k(\eta,t) = 0. \tag{5}$$

where $F(\eta)$ may be conveniently chosen to make $k(\eta,t)$ simple. This can be shown by applying equation (2) to (1) and integrating by parts. In eq. (4), g(t) is found from

$$g(t) = e^{\int \frac{b(t) - a'(t)}{a(t)} dt} \tag{6}$$

which makes the differential operator of eq. (1) self-adjoint. That is

$$g(t)\left[a(t)q''(t) + b(t)q'(t)\right] = \left[p(t)q'(t)\right]'. \tag{7}$$

III. The Inverse Transform

With the direct transform eq. (2), known, it is also possible to derive explicit formulas for the inverse transform

$$T^{-1}\left[Q(\eta)\right] = q(t). \tag{8}$$

The inverse transform is sometimes essential; particularly in case that development of a generally applicable transform table becomes excessively tedious and laborious due to the complex form of the transform kernel.

One may assume that the inverse transform is essentially unique, since one can always prove the uniqueness for each particular transformation. This means that a given $Q(\eta)$ cannot have more than one inverse form $q(t)$ that is continuous at almost all positive t. Note that not every function of η is a transform of some $q(t)$. Therefore, in addition to the uniqueness, one must also consider the conditions imposed on $q(t)$ and $Q(\eta)$.

The method of finding the inversion formula is based on a method used in the development of the complex inversion integral for the Laplace transformation[4].

The transform $Q(\eta)$ is assumed to be a function of the complex variable η, analytic in the half plane $R(\eta) \geq \sigma$ and of the order of η^{-k} as $\eta \to \infty$ $(k > 0)$.

When the conditions imposed on $Q(\eta)$ are satisfied, $Q(\eta)$ may be expressed in terms of its values along a vertical line by the line integral as defined below:

$$Q(\xi) = \lim_{\beta \to \infty} \frac{1}{2\pi j} \int_{\sigma - j\beta}^{\sigma + j\beta} \frac{Q(\eta)}{\xi - \eta} \, d\eta \tag{9}$$

where ξ is also a complex number and $R(\xi) > \sigma_a$. Equation (9) is simply Canchy's integral formula in the complex plane.

Now, apply the inverse transformation as defined by eq. (8) to the function of ξ on both sides of eq. (9); then

$$T^{-1}[Q(\xi)] = T^{-1}\left[\frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \frac{Q(\eta)}{\xi - \eta} \, d\eta\right]. \tag{10}$$

If the order of the inverse transform operator T^{-1} and the integration along the line $R(\eta) = \sigma$ can be interchanged, then one will have,

$$T^{-1}[Q(\xi)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} Q(\eta) \, T^{-1}\left[\frac{1}{\xi - \eta}\right] d\eta. \tag{11}$$

Let $Q(\xi)$ be the T transform of $q(t)$ as defined by eq. (2). The inverse transform to ξ of $\frac{1}{\xi-\eta}$ may be defined as

$$T^{-1}\left[\frac{1}{\xi-\eta}\right] = i(\eta,t). \tag{12}$$

Substituting (12) and $T^{-1}[Q(\xi)] = q(t)$, eq. (8), into (11), there results

$$q(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} Q(\eta)\, i(\eta,t)\, d\eta. \tag{13}$$

Equation (13) is the inversion formulas of the T transformation. $i(\eta,t)$ may be called the inverse kernel.

Equation (12) defines the inverse transform kernel $i(\eta,t)$. Therefore, once one knows the inverse transform to ξ of $\frac{1}{\xi-\eta}$ for a particular transformation, one automatically defines the inversion formula of this transformation.

The key step in this method of finding the inversion formula lies in the step from eq. (10) to eq. (11), where an interchange of the inverse transform operator T^{-1} and the integration along the line $R(\eta)=\sigma$ must be performed. In suggesting this method of finding the inversion formula, it has been, in fact, assumed that the inversion formula takes the form of eq. (13). If for a particular transformation, this assumption is valid, i.e., there exists an inversion formula of the form of eq. (13), then proof of the validity of interchange of the order of the operators involved in eq. (10) becomes a rather trivial matter.

If one applies the direct transformation of the function of ξ to both sides of equation (12) and uses equations (2) and (8), one will have

$$\int_0^\infty i(\eta,t)\, h(\xi,t)\, d\xi = \frac{1}{\xi-\eta} \tag{14}$$

where $h(\xi,t)$ is the same as $h(\eta,t)$, the known direct transform kernel, except substituting η by ξ. Equation (14) suggests that for a particular transformation whose inverse kernel is defined by eq. (12), the inverse transform kernel, $i(\eta,t)$, may be a solution of this integral equation.

To summarize, the direct transformation and inverse transformation formulas of the generalized T transformation are as follows:

$$Q(\eta) = \int_0^\infty q(t)\, h(\eta,t)\, dt \qquad \sigma > \sigma_a \tag{15}$$

$$q(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} Q(\eta)\, i(\eta,t)\, d\eta \qquad t \geq 0. \tag{16}$$

In checking the validity of eq. (12) or eq. (14), one finds that the direct and inverse kernels of the Laplace transformation pair do satisfy these equations, thus:

$$\int_0^\infty e^{\eta t} e^{-\xi t} dt = \frac{1}{\xi-\eta} \qquad \xi > \eta. \tag{17}$$

The transform kernels of the transformation pair for the Cauchy network (modified Mellin transform[5]) as developed by Aseltine[1] satisfy equation (14) as follows:

$$\int_{t_0}^\infty \left(\frac{t}{t_0}\right)^{-\eta} \frac{t^{\xi-1}}{t_0^{\xi}}\, dt = \frac{1}{\eta-\xi} \qquad \xi < \eta. \tag{18}$$

Note that the analytic region defined by this transformation is $\xi < \eta$. For the Bessel network, the transform pair given by Aseltine[1] is a modified form of the Meijer transform[6]. The inversion formula of the Meijer transform with the inverse kernel $\sqrt{2\pi\eta t}\, I_\nu(\eta t)$ is defined to recover a function $f(t)$ for the entire range of t. If one desires a direct Laplace analog for the Bessel network, the inverse kernel which contains only part of the Meijer inverse kernel may be determined from eq. (14).

The consideration of the development of inversion formula provides much deeper understanding of the transform method for time-varying networks. By the procedure discussed above, it is clear that transformation pairs directly analogous to the Laplace transformation pair may be developed for both Cauchy and Bessel networks (with suitable modification of the lower limits of the direct transformation integral). It is not necessary to obtain the transformation pair by modifying the Mellin and Meijer transforms.

IV. Qualitative Considerations of Transformation Method

The main advantage of using a transform method in the solution of the differential equation representing a linear time-varying network is that the differential equation can be handled algebraically. The technique involved is to convert eq. (1) into eq. (3) by the application of the integral transformation as defined by eq. (2). In order to make this application possible, the transformation must be capable of handling a large class of driving functions and the differential operators of the network equation. These fundamental requirements are important from the point of view of the practical application of the transformation method to time-varying networks. In view of eq. (2), the limitations on the character of the function $q(t)$ and on the range of the variable of η depend obviously on the character of the transformation kernel $h(\eta,t)$.

Among the driving functions most commonly used in the analytical solutions of physical systems, such as the electric networks, are the sectionally continuous functions. An example of this type of functions is shown below

$$q(t) = \begin{cases} 0 & t < t_1 \\ a & t_1 < t < t_2 \\ 0 & t_2 < t \end{cases} . \tag{19}$$

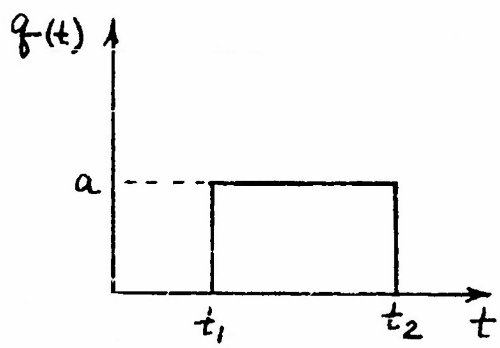


Figure 1

Any discontinuities of such a function in the interval (t_1, t_2) must be a finite jump. This class of functions includes the important unit step function defined as

$$\mathcal{U}(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t \,. \end{cases} \tag{20}$$

For a transformation developed for a particular time-varying network, if the character of its transform kernel is such that the transformation is not capable of handling the unit-step function and other more general functions, the transformation method is probably of little use for the solutions of the differential equation of that particular network. Therefore, it is possible to set up some basic requirements that a transformation must fulfill. For the functional transformations, it is very convenient to choose the transformation of the unit step function as a measure of the capability of a transform kernel. This leads to a rather simple consideration of the existence of the transformation integral:

$$U(\eta) = \int_0^\infty h(\eta,t)\,dt \tag{21}$$

where $U(\eta)$ is the transform of the unit step function.

The transformation of the differential operators of eq. (1) is performed in a manner somewhat different from that of the ordinary Laplace transformation. Since the transform kernel is developed to make the differential operators self-adjoint, the transformation of the differential operators is performed in the following manner[1]:

$$\begin{aligned} &T\left[a(t)\, q''(t) + b(t)\, q'(t)\right] \\ &= F(\eta)\, Q(\eta) + g(\eta)\, q'(0+) + j(\eta)\, q(0+), \end{aligned} \tag{22}$$

In order that the transformation method be of practical use, equation (22) must be satisfied. So the initial conditions are brought into the transformation. Equation (22) gives the fundamental operational property of the "T" transformation, the property that makes it possible to replace the differential operation by a simple algebraic operation. Note that the differential operators may not be decomposed or factored.

V. Existence of Transformation

Thus far both the development of a transformation pair for a linear time-varying network whose behavior is described by eq. (1), and the fundamental properties that the transformation must possess, have been considered. Now, it is desired to know the limits on the applicability of the transformation method to time-varying networks. It is obvious that with the time-varying coefficients of eq. (1) unrestricted, a satisfactory transformation may not exist at all. Thus, it is important to settle what kind of network equations do, and what kind of network equations do not, permit the use of transformation method.

From the preceding section, a useful transformation must be one which is capable of handling the class of sectionally continuous functions (unit step function is actually considered) and the differential operator of the network equation. Reconsider the "T" transformation of a unit step function

$$U(\eta) = \int_0^\infty h(\eta, t)\,dt. \qquad (21)$$

In order that the infinite integral on the right-hand side of equation (21) converges uniformly, it is necessary that the transform kernel $h(\eta,t)$ be continuous and bounded at both upper and lower limits for all values of η in some interval (α,β). Since $h(\eta,t)$ is defined in eq. (4) as the product of $g(t)$ and $k(\eta,t)$, eq. (21) becomes

$$U(\eta) = \int_0^\infty g(t)\, k(\eta,t)\, dt. \tag{23}$$

Following a well-known test, it is convenient to establish the following conditions for the uniform convergence of the infinite integral of equation (23).

1. $k(\eta,t)$ is continuous for $0 < t < \infty$.
2. $g(t)$ is continous for $0 < t < \infty$.
3. The product of $g(t)$ and $k(\eta,t)$ is bounded at both upper and lower limits of the integral for all η in some interval (α,β).

Note that it is not necessary for each individual function of $g(t)$ or $k(\eta,t)$ to be bounded at both upper and lower limits; but the product of $g(t)$ and $k(\eta,t)$ must be bounded at both limits.

The above three conditions will afford one an opportunity to relate singular points of the network equation to its transform kernel, thus permitting one to settle what kind of system equation permits the use of the transformation method. To facilitate further discussion, re-establish the kernel equation in the following manner:

$$k_{tt}(\eta,t) + \varphi(t)\, k_t(\eta,t) + \theta(t)\, k(\eta,t) = 0 \tag{24}$$

where $\varphi(t)$ corresponds to $\frac{b(t)}{a(t)}$ in equation (5) and $\theta(t)$ to $\frac{1}{a(t)}$ in the same equation.

$k(\eta,t)$, then, is a solution of equation (24). For a particular network equation, the question as to whether a satisfactory $k(\eta,t)$ as defined in condition I exists or not may be settled by considering the singular points of the kernel equation. In eq. (24), $F(\eta)$ is considered a function of fixed parameter η. Considering the point t_o and its neighborhood, it is well known from the fundamental theory of ordinary differential equations[7,8] that: (a) if $\varphi(t)$ and $\theta(t)$ are continuous and analytic functions of t in this neighborhood, a unique solution, continuous and analytic in this same neighborhood can be determined for eq. (24); (b) if either $\varphi(t)$ or $\theta(t)$ or both possess a regular singular point* at t_o, eq. (24) has two regular integrals in the neighborhood of this point; (c) if either $\varphi(t)$ or $\theta(t)$ or both possess an irregular singular point* at t_o, eq. (24) cannot have two regular integrals in the neighborhood of this point. But there may be one regular integral or there may be none.

---

*Fuchs' theorem[9] in terms of eq. (24): If $\varphi(t)$ or $\theta(t)$ or both of eq. (24) possess a singular point at $t=t_\ell$ and if this singular point of $\varphi(t)$ and $\theta(t)$ is removable by multiplying $\varphi(t)$ and $\theta(t)$ by the factors $(t-t_\ell)$ and $(t-t_\ell)^2$ respectively, then the equation has two integrals (in terms of convergent development of power series solution) in the neighborhood of t_ℓ. A singular point of this type is a regular singular point, otherwise an irregular singular point.

To decide whether a regular integral exists or not, it is most convenient to consider the indicial equation of the power series solution developed about a singular point. Note that the above rules are applicable to the point at infinity only if a transformation which transforms the point at infinity to the origin is performed.

As a rule, therefore, solution of eq. (24) may be found in the neighborhood of a singular point as well as at an analytic point. In general, the two independent solutions may be expressed in terms of power series about a point where one desires to expand them. These solutions are only valid within their convergence-circles, centered at this point, and whose radii are equal to the absolute value of the distance between this point and the nearest singular point. It becomes clear, then, that for a given kernel equation of the type of eq. (24), the question of whether a desired $k(\eta, t)$ continuous for $0 < t < \infty$ exists or not may be settled by simply considering the number, nature and locations of singular points of the equation.

From the above consideration, one may conclude that for a kernel equation possessing one or two singular points, a desired $k(\eta, t)$, relatively simple in form, may exist. On the other hand, for the kernel equation of more complex form, a solution, continuous for $0 < t < \infty$ may still be possible by applying the technique of analytic extension in the solution of the equation. But the mathematical manipulations would be very difficult, and the resultant $k(\eta,t)$ would be very complicated and often beyond the possibility of practical development of the transformation. In general, it might be more profitable to use a solution which is only continuous for $t_\ell < t < \infty$ where t_ℓ is the nearest singular point of the kernel equation to the point of infinity. This, of course, involves a change of the lower limit of the direct transformation integral from zero to a finite value of t_o $(t_o > t_\ell)$, or one may even define a direct transformation integral with finite limits. Finding $k(\eta,t)$ is the most difficult step in the development of the transform kernel. Therefore, a knowledge of whether a suitable $k(\eta,t)$ is obtainable from the kernel equation is a great help in deciding the use of transformation method before becoming involved in the actual solution of the equation.

$g(t)$ of condition 2 may be redefined as follows

$$g(t) = \theta(t)\, e^{\int \varphi(t)\,dt} \tag{25}$$

Equation (25) shows that infinite discontinuities due to the presence of singular points in $\varphi(t)$ or $\theta(t)$ or both may appear in $g(t)$ unless the effect is cancelled out due to the mathematical operation on the right-hand side of eq. (25) or due to some conditions imposed on certain parameters in $\varphi(t)$ or $\theta(t)$. In general, finding $g(t)$ is a straightforward process. Therefore, whether a suitable $g(t)$ exists or not may be settled without difficulty.

When the consideration of singular points of the kernel equation indicates that desired $k(\eta,t)$ and $g(t)$ as defined in conditions 1 and 2 respectively do exist, then one may proceed to evaluate $k(\eta,t)$ from the kernel equation. To insure the boundedness of $h(\eta,t)$ at the upper limit, one chooses from the two independent solutions of equation (24) the one which makes $k(\eta,t)$ vanish rapidly as $t \to \infty$ if there exists such a solution. The next step is to perform the product of $g(t)$ and $k(\eta,t)$ to obtain $h(\eta,t)$ and to check the condition 3 which requires $h(\eta,t)$ to be bounded at both upper and lower limits of the transformation integral for all η in some interval (α,β). Note that when η is considered a complex variable, $h(\eta,t)$ is to be bounded at both limits for all $R(\eta)$ in some real interval (α,β).

In some cases, the boundedness of $h(\eta,t)$ or $g(t)k(\eta,t)$ may not be obvious. Many known mathematical methods[5] are available for tests of the uniform convergence of the transformation integral eq. (21) or eq. (23). In many cases modification of the transformation integral, such as the limit of the integral, or imposing additional conditions on certain parameters in either $g(t)$ or $k(\eta,t)$ or both, is absolutely necessary in order to insure the boundedness of $h(\eta,t)$.

Thus far, the most difficult part in the process of developing a transform for a particular time-varying network has been considered. The remaining basic requirement to be satisfied by a particular transform is its capability of the transformation of the differential operators of the network equation. This can best be done by the direct integration of the left-hand side of eq.

(22) by substituting $h(\eta,t)$ and the differential operators of a particular network equation into the integral. Again, in some cases, a slight modification on the transformation integral's limits might be required in order to satisfy the right-hand side of eq. (22). But, one does not wish to impose any condition on $q(t)$ other than those on $v(t)$, the driving function.

To complete the development of transformation, one may then derive the inverse transform by the method indicated in Section III and discuss the limitations on the character of the function $v(t)$ and on the range of the variable η.

For example, consider a time-varying network with the following behavior,

$$\frac{1}{t^2} q''(t) - \frac{1}{t^3} q'(t) + \frac{1}{C_o} q(t) = v(t) \tag{26}$$

The transformation pair obtained is as follows:

$$Q(\eta) = \int_{t_o}^{\infty} q(t)\, t\, e^{-\eta(t^2 - t_o^2)}\, dt \qquad \sigma > \sigma_a \tag{27}$$

$$q(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} Q(\eta)\, 2 e^{\eta(t^2 - t_o^2)}\, d\eta \qquad t > t_o \tag{28}$$

which is an analogy to the Laplace transform. The reason for the modification of the lower limit of the direct transform integral is obvious since the network equation has a singular point at $t = 0$. The direct transform kernel $t\, e^{-\eta(t^2 - t_o^2)}$ is now continuous for $t_o < t < \infty$ and bounded at both upper and lower limits of the integral defined by eq. (27). With little difficulty, one may show that this transform is capable of transforming both unit step function (for $t > t_o$) and the differential operators of the network equation.

REFERENCES

1. Aseltine, J. A., Transform for Linear Time-Varying Systems, University of California, Department of Engineering, Los Angeles, Report No. 52-1, February, 1952.

2. Aseltine, J. A., Davis, H., Ho, E. C., Trautman, D. L., Extension of the Operational Calculus to Time-Varying Networks, paper presented at AIEE Summer Meeting, Los Angeles, June 23, 1954.

3. Ho, E. C., University of California, Los Angeles, Network Research Bulletins Nos. 4 and 6, 1952.

4. Churchill, R. V., Modern Operational Mathematics in Engineering, McGraw-Hill, 1944.

5. Titchmarsh, E. C., Theory of Fourier Integrals, Oxford, 1948.

6. Meijer, C. S., "Über eine Erweiterung der LaPlace-Transformation," I, II, Nederl. Akad. Wetensch., proc., 43 (1940) pp. 599-608, 707-711.

7. Ince, E. I., Ordinary Differential Equations, Dover.

8. Morse, P. M., and Feshbach, H., Methods of Theoretical Physics, McGraw-Hill, 1954.

9. Margenau, H., and Murphy, G. M., The Mathematics of Physics and Chemistry, Van Nostrand, 1943.

GENERALIZED OPERATIONAL CALCULUS FOR TIME-VARYING NETWORKS

Part II

Harold Davis

Part II of this report concerns (a) those integral transformations which generate an operational calculus for general linear differential operators, and (b) the application of such transformations to the analysis and synthesis of a class of linear time-varying networks.

1. Introduction

Much of present day fixed parameter linear network theory is based on the properties of the Laplace transform and its relation to the differential operator d/dt. To the extent that this is essentially all that is involved, it follows that fixed parameter network theory can be embedded in a formal linear network theory if the existance of a suitably generalized Laplace transform is postulated. For example, suppose that L is a differential operator,

$$Ly \equiv a_N(t)[d^N/dt^N]y(t) + \cdots + a_0(t)y(t), \quad 0 \leq t \leq \infty \qquad (1)$$

and $k(t,\lambda)$ is a solution to

$$L^*y \equiv (-1)^N[d^N/dt^N]a_N(t)y(t) + \cdots + a_0(t)y(t) = \lambda y(t) \qquad (2)$$

(L^* is the formal adjoint of L). Suppose further that $k(t,\lambda)$ is the kernel of a one-to-one integral transformation (on a suitable, complete class of functions), say,

$$F(\lambda) = \int_0^\infty k(t,\lambda)\, f(t)\, dt \qquad (3)$$

It would then follow that for Lf in this class,

$$\int_0^\infty k(t,\lambda)\, Lf(t)\, dt = \lambda \int_0^\infty k(t,\lambda)\, f(t)\, dt + B(k,f) \qquad (4)$$

If $k^{[p]}(t,\lambda)\, f^{[q]}(t)$ vanish as $t \rightarrow \infty$, $0 \leq p,q \leq N$ then B is a bilinear form in $f(0), f'(0), \ldots f^{[N-1]}(0)$ and a set of N functions of λ not depending upon $f(t)$. To make the picture complete we would need some form of Parseval's equality. It is expected that, formally, such should exist. Several transforms satisfying all the above requirements have been discussed by Aseltine[1].

The following material is concerned first with a discussion of the application of transforms of the type described above to the analysis and synthesis of a class of linear time-varying networks. Secondly, it is pointed out that for first order differential operators such integral transforms exist. The material concludes with a brief mention of Coddington's theorem on generalized Fourier transforms.

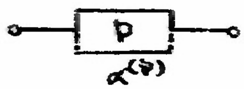
II. On the Formalities of a Generalized Network Theory

We first ask what general features or structures of linear networks can be studied by application of an integral transformation of the form described above. The answer is fairly immediate. Consider network elements such that when a voltage e(t) is applied, the current i(t) that flows is related to e(t) by,

$$\alpha^{(p)} L^{p}\, i(t) = e(t)\ ;\ -m \leq p \leq n\ ;\ 0 \leq m,n < \infty \qquad (5)$$

$$\left(L^{p+1} y \equiv L(L^{p} y) \right)$$

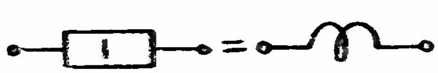
where $\alpha^{(p)}$ is a real number, L a linear differential operator (of the type considered above) and p an integer. In illustrating network structures let us portray these elements as follows:

SYMBOL	EQUATION
	$\alpha^{(p)} L^p i = e$

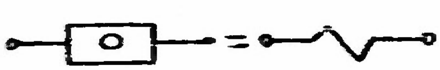
It will be convenient to distinguish the case where $\alpha^{(p)}$ takes on only non-negative values. We shall refer to this as the case of <u>positive elements</u>.

Example:

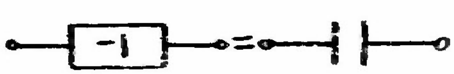
Let $a(t) = 1$, $b(t) = 0$; then $L = {}^{d}/dt$ and we have the case of conventional fixed parameter elements.



$\alpha^{(1)}$ = self inductance

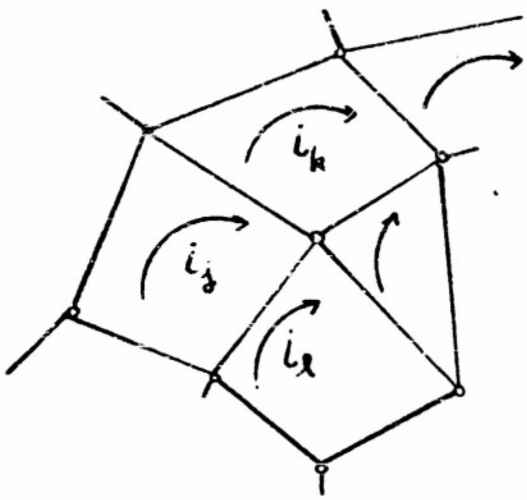


$\alpha^{(0)}$ = resistance



$\alpha^{(-1)}$ = elastance (reciprocal of capacitance).

Now consider any interconnection of the various types of time-varying elements discussed above. Such a network will be a node-to-node connection of branches as indicated in Figure 1. Every branch can be assumed to be a series connection of the various types of elements, and a voltage or current source. We need only consider one of each element type as being present, since for each type a series connection of elements is equivalent to a single element of that type. We will consider the sources to be voltage sources so that we can set up a loop current analysis. Of course any or all of the coefficients or voltages can vanish.



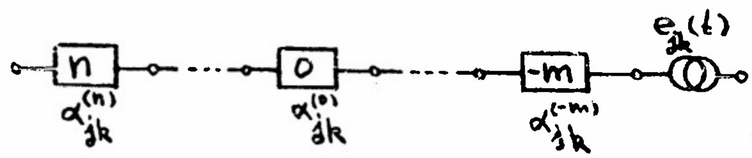


Figure i

By assuming an appropriate choice of mesh currents, and summing loop voltages to zero in every loop we get a system of linear integro-differential equations in the mesh currents,

$$\sum_k \gamma_{jk} i_k(t) = e_j(t) = \sum_k e_{jk}(t) \tag{6}$$

where γ_{jk} is an integro-differential operator of the form,

$$\gamma_{jk} = \alpha_{jk}^{(n)} L^n + \cdots + \alpha_{jk}^{(-m)} L^{-m} \tag{7}$$

In setting up these differential equations we can always incorporate the initial conditions in the various driving voltages $e_j(t)$. That is to say, if we suitably modify the assumed drives $e_j(t)$, the values of

$$L^{-m} i\big|_{t=0},\ d/dt\, L^{-m} i\big|_{t=0}, \ldots,\ d^{N(n+m)-1}/dt^{N(n+m)-1}\, L^{-m} i\big|_{t=0}$$

will be zero for all loop currents. Let us assume the equations have been set up in this way. Then, taking the transform of these equations by that integral transformation which is associated with the operator L as described above, we have,

$$\sum_k Z_{jk} I_k(\lambda) = E_j(\lambda) \qquad (8)$$

where

$$Z_{jk} = \alpha_{jk}^{(n)} \lambda^n + \cdots + \alpha_{jk}^{(0)} + \cdots + \alpha_{jk}^{(-m)} \lambda^{-m} \qquad (9)$$

and $E_j(\lambda)$ is the transform of $e_j(t)$, and $I_j(\lambda)$ the transform of $i_j(t)$.

This is the functional form of the loop equations in the transform domain. It is the form which is so familiar in the fixed parameter case. To proceed from here to the point of a general theory of analysis and synthesis of networks, we need only ape the steps taken in the case of networks of fixed parameter elements. That is, we replace the parameters of the Laplace transform by the λ of our generalized transform; replace inductor by "type I element" and so on. Impedance (or impedance function) is again defined as

$$Z(\lambda) = \frac{E(\lambda)}{I(\lambda)}$$

Admittance, image parameters, and general circuit parameters of four terminal networks follow in the same way.

III. A Special Case

In the special case where only positive elements are present, and these are characterized by either L^1, L^0, or L^{-1}, and where L itself is a first order operator, the theory is quite satisfactory and complete. The facts of the matter are as follows: First, as we shall see, the formalism or an integral operator can be made precise under fairly general conditions on the operator L. Secondly, most of the theorems of the present fixed parameter network theory do not depend on the constancy of inductance and capacitance with time. Rather, they depend on the existance of three types of (positive) elements characterized by L^1, L^0, and L^{-1} where $L = {}^{d}/dt$ (a great deal of network theory omits from consideration the notion of mutual impedance).

To make this more explicit we will suggest a sort of principle for inventing statements of generalizations of certain standard network theorems. In fixed parameter network theory, there are a great many theorems relating the physical structure of a network to the functional form of impedance type functions. In these theorems, one identifies the impedance Z(s) = s with an inductor, $Z(s) = s^{-1}$ with a capacitor, and so on. Actually, this identifies element types through the exponent of the basic operator associated with fixed parameter networks, namely $^{d}/dt$. This identification can be considered as the basic bridge between the physical specification of the network, and the mathematical specification of the impedance function of that network. Now, we suggest that any network theory which depends only on this identification of element types, and on the theory of analytic (impedance) functions will have a valid extension if we simply translate it into the appropriate "generalized operator" language.

As a caution in this regard, no theorem requiring the use of coupled coils (non-ideal transformers) can be extended, since the coupled coil involves more than the two considerations mentioned in the last paragraph.

Applying the above principle, we get a generalization of Foster's Reactance Theorem for time-varying elements, the canonical forms for two element kind networks, and so on. Such theorems being the very foundation of fixed parameter network theory, we thus have considerably broadened the basis for the analysis and synthesis of the class of networks considered here. (For details of the application of these theorems in the classical fixed parameter case see, e.g., Guillemin's "Communication Networks."[8])

Possibly the most significant theorem is the following:

"A necessary and sufficient condition that an analytic function of the complex parameter λ represent the impedance of some two terminal network of the three types of elements considered in this section, is that this function be positive real; that is, that it be an analytic function which is real when λ is real, and whose real part is positive when the real part of λ is positive."

The necessity of the condition can be proven by following a slight modification of an argument of Brune[4] applied to fixed parameter networks. However, Brune's proof of the sufficiency of the condition for fixed parameter networks is not adequate for our purposes, since he used coupled coils whenever necessary. We can, however, call on the method of Bott-Duffin[5] which depends only on the ability to identify the three types of elements assumed.

As a simple example of network synthesis using elementary methods consider the following problem: Suppose that we are required to find a two terminal network such that, when the driving voltage is $e(t) = \delta(t)$, where $\delta(t)$ is the Dirac impulse "function"; the current which results will be

$$i(t) = ((1+t)^{1/2} - 2)/2(1+t)^{3/2} \qquad (10)$$

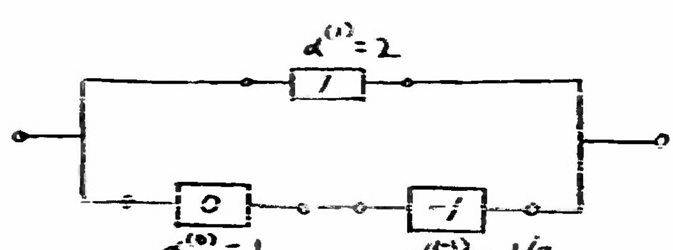
We ask for a network of elements characterized by powers of the operator $L = (1+t)\,{}^{d}/dt + 1$. Thus, $k(t, \lambda) = (1+t)^{-\lambda}$. If we compute the transforms of the voltage and current by the transformation associated with this operator L, we shall find their ratio to be

$$Z(\lambda) = \frac{E(\lambda)}{I(\lambda)} = (2\lambda + 4\lambda^2)(1 + 2\lambda + 4\lambda^2)^{-1} \qquad (11)$$

Expanding the admittance $Y(\lambda) = [Z(\lambda)]^{-1}$ into partial fractions we have,

$$Y(\lambda) = (2\lambda)^{-1} + 2\lambda(1+2\lambda)^{-1} \qquad (12)$$

Now let us apply the principle of identification mentioned before. We know that if this were the case of fixed parameter synthesis with λ replaced by the conventional $s = \sigma + i\omega$ the required network would be an inductance of 2 henries, in parallel with a series connection of a resistance of 1 ohm and a capacitance of 2 farads. Since inductance is identified with a plus 1 exponent, resistance a zero exponent, and capacitance with a minus 1 exponent we can immediately write down the circuit with proper element types and correct values of the coefficients. Thus, we immediately have the solution indicated in the following diagram.



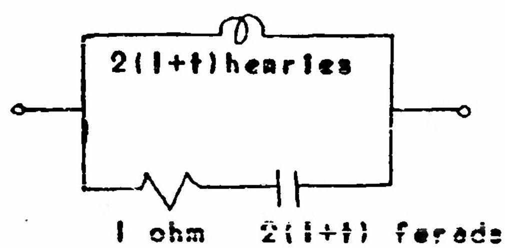
As a consequence of the fact that L is a first order differential operator, the element characterized by the exponent 1 can be considered as an ordinary inductor whose inductance is (1 + t), for then

$$Li \equiv [(1+t)d/dt + 1]i(t) = d/dt[(1+t)i(t)] = e(t)$$

as is required. Similarly the -1 element can be considered as an element with capacitance (1 + t) so that

$$Le \equiv [(1+t)d/dt + 1]e(t) = d/dt[(1+t)e(t)] = i(t)$$

as required. Thus, the above solution can be redrawn as follows:



To summarize the case of positive elements characterized by a first order differential operator taken to powers 1, 0, and -1 only, we obtain time-varying generalizations of most of the more powerful theorems of fixed parameter network theory.

IV. The Steady-State Impedance Concept

It is worth calling attention to another aspect of what might be called the transform domain approach. Let us fix our attention on networks of elements characterized by powers of a differential operator L as before. If we drive a single loop of such elements with a voltage proportional to the function $k(t,\lambda)$ (the kernel of the transform associated with L), then the resultant current will also be proportional to this same function but will differ by a multiplicative factor - the value of the λ-domain impedance of the loop at the value λ. We must expect this since we are driving the circuit with a function that bears the same relation to these network elements as sinusoids do to fixed parameter elements. This suggests that the conventional steady-state physical interpretation of impedance defined as the ratio of voltage across to current through a two terminal network when driven by a single frequency carries over by simply thinking of functions of the form $k(t,\lambda)$ as generalized sinusoids and of λ as a generalized frequency.

V. Functional Transforms Associated with First Order Linear Differential Operators

We now discuss briefly the connection between a first order linear differential operator and an associated generalized Laplace transform.

Throughout the following we shall restrict ourselves to functions defined on the half line, $0 \leq t \leq \infty$.

Let $a(t)$ and $b(t)$ be functions such that $b(t)$ and da/dt are defined and finite, and that $a(t) \neq t$ for all $t \geq 0$. Consequently, either $a(t) > 0$ or else $a(t) < 0$ holds for all $t \geq 0$. Without loss of generality, we may assume that $a(t) > 0$, and further, that $a(0) = 1$.

Suppose also that $a(t)$ and $b(t)$ are such functions that there exist numbers A and B such that,

$$(1/t)\int_0^t [a(u)]^{-1}du > A > 0 \tag{13}$$

and

$$(1/t)\int_0^t |b(u)[a(u)]^{-1}|du < B < \infty \tag{14}$$

Now, we define the linear first order differential operator L by

$$Ly \equiv [a(t)d/dt + b(t)]y(t) \tag{15}$$

The adjoint operator L^* is defined by,

$$L^*y \equiv [-(d/dt)a(t) + b(t)]y(t) \tag{16}$$

Let λ be a complex parameter, and $k(t,\lambda)$ be the solution to the equation,

$$L^*y = \lambda y \; ; \; y(0) = 1 \tag{17}$$

or written out

$$-a\,dy/dt + y(b - da/dt) = \lambda y \; ; \; y(0) = 1$$

Correspondingly, let $K(t,\lambda)$ be the solution to the equation,

$$Ly = \lambda y \; ; \; y(0) = 1 \tag{18}$$

Because L is only of the first order, we can write these solutions in terms of a single quadrature. Explicitly,

$$k(t,\lambda) = [a(t)]^{-1} \exp\left(\int_0^t [b(u) - \lambda][a(u)]^{-1} du \right) \tag{19}$$

$$K(t,\lambda) = \exp\left(- \int_0^t [b(u) - \lambda][a(u)]^{-1} du \right) \tag{20}$$

We have the following:

Theorem

Let $f(t)$ be a function such that for some c, $0 < c < \infty$,

$$\int_0^\infty |f(t)| [a(t)]^{-1} e^{-ct} dt < \infty \tag{21}$$

Then,

$$F(\lambda) \equiv \int_0^\infty k(t,\lambda) f(t) dt \tag{22}$$

is an analytic function of λ in the half plane $\mathrm{Re}\,[\lambda] > \frac{c + B}{A}$.

Furthermore,

$$f(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} K(t,\lambda) F(\lambda) d\lambda \; ; \; \beta > \frac{c+B}{A} \tag{23}$$

at every point of continuity of $f(t)$.

We also have that if $L^n f(t)$, $n \geq 1$, also satisfies the same sort of condition as f above, then,

$$\int_0^\infty k(t,\lambda) L^n f(t)\,dt = \lambda^n \int_0^\infty k(t,\lambda) f(t)\,dt +$$

$$-\lambda^{n-1} f(0) - \lambda^{n-2} Lf\big|_{t=0} - \cdots - L^{n-1} f\big|_{t=0} \qquad (24)$$

If $F(\lambda)$ and $G(\lambda)$ are respectively the transforms of $f(t)$ and $g(t)$, and F, G are analytic for $\mathrm{Re}\,[\lambda] \geqq -\delta$, $\delta > 0$, then, if $|\beta| < \delta$,

$$(2\pi i)^{-1} \int_{\beta - i\infty}^{\beta + i\infty} F(\lambda) G(\lambda)\,d\lambda =$$

$$= \int_0^\infty [a(t)]^{-1} \exp\left(2\int_0^t [a(u)]^{-1} b(u)\,du\right) f(t) g(t)\,dt \qquad (25)$$

Here are two examples of such integral transforms.

1. Let $a(t) = 1$, $b(t) = 0$. Then $L^* y = \frac{dy}{dt}$. $Ly = \frac{dy}{dt}$, and so $k(t, \lambda) = e^{-\lambda t}$, $K(t, \lambda) = e^{\lambda t}$. This is the case yielding the conventional Laplace transform.

2. Let $a(t) = 1 + t$, $b(t) = 0$. Then $k(t, \lambda) = (1 + t)^{\lambda - 1}$, and $K(t, \lambda) = (1 + t)^{-\lambda}$. This yields a modification of the Mellin Transform. Specifically, the image of a function under this transformation is identical with the image under the Mellin Transformation of the same function, but translated one unit to the right on the t axis.

The first example is the simplest, but most instructive. It shows how the Laplace Transform fits into the scheme. The second example exhibits a bonafide varying operator.

Now suppose $f(t)$ is as required in the theorem, that is, for some finite and real number c,

$$\int_0^\infty |f(t)|\,[a(t)]^{-1} e^{-ct}\,dt < \infty \qquad (26)$$

Then,

$$|F(\lambda)| = \left|\int_0^\infty k(t,\lambda) f(t)dt\right| \leq \int_0^\infty |k(t,\lambda)|\,|f(t)|dt =$$

$$= \int_0^\infty [a(t)]^{-1} \exp(-Re[\lambda\int_0^t a^{-1} du] + \int_0^t b/a\, du)\, |f(t)| dt \qquad (27)$$

But

$$\int_0^t [a(u)]^{-1} du \quad > At,$$

and

$$\left|\int_0^t b/a\, du\right| \leq \int_0^t |b/a| du < Bt,$$

by hypothesis. Hence,

$$|F(\lambda)| \leq \int_0^\infty [a(t)]^{-1} \exp(-Re[\lambda At] + Bt)|f(t)| dt \qquad (28)$$

By reason of the hypothesis on f(t), the latter integral is convergent for all λ for which

$$Re[\lambda] \geq \frac{c+B}{A}$$

In fact it is clear that convergence is uniform for λ

$$Re[\lambda] \geq \frac{c+B}{A}$$

Hence, in this half plane, $F(\lambda)$ is an analytic function of λ and has no singularities there.

Now write,

$$g(t) = \lim_{\alpha\to\infty} (2\pi i)^{-1} \int_{\beta - i\alpha}^{\beta + i\alpha} K(t,\lambda) F(\lambda) d\lambda =$$

$$\lim_{\alpha\to\infty} \int_{\beta - i\alpha}^{\beta + i\alpha} K(t,\lambda) d\lambda \int_0^\infty k(\tau,\lambda) f(\tau) d\tau \quad ; \beta > \frac{c+B}{A} \qquad (29)$$

Since the integral for $F(\lambda)$ is uniformly convergent for all λ such that

$$\operatorname{Re}\lambda \geq \frac{c+B}{A}$$

we may interchange orders of integration, and

$$g(t) = \lim_{d\to\infty} \int_0^\infty f(\tau)d\tau\,(2\pi i)^{-1} \int_{\beta-id}^{\beta+id} K(t,\lambda)\,k(\tau,\lambda)\,d\lambda \qquad (30)$$

Now,

$$\int_{\beta-id}^{\beta+id} K(t,\lambda)k(\tau,\lambda)\,d\lambda = \frac{1}{a(\tau)} \int_{\beta-id}^{\beta+id} \exp\left(-\int_t^\tau \frac{\lambda-b(v)}{a(v)}dv\right)d\lambda =$$

$$= 2i\,(a(\tau)\varphi(\tau,t))^{-1}\exp(\psi(\tau,t)-\beta\varphi(\tau,t)\sin(d\varphi(t,\tau)) \qquad (31)$$

where,

$$\varphi(\tau,t) = \int_t^\tau (a(v))^{-1}dv \quad , \quad \psi(\tau,t) = \int_t^\tau (b(v)/a(v))\,dv$$

For every fixed t, both φ and ψ are continuous functions of τ : both φ and ψ vanish when $\tau = t$; and φ is a strictly increasing function of τ for $-\infty < \tau < +\infty$. Thus φ has a single valued continuous, strictly increasing inverse,

$$\varphi_t^{-1}[\varphi(\tau,t)] = \tau \qquad \text{and} \qquad \varphi_t^{-1}[0^+] = t^+ , \; \varphi_t^{-1}[0^-] = t^-$$

Thus, for each fixed t,

$$g(t) = \lim_{d\to\infty} \int_0^\infty f(\varphi_t^{-1}[\varphi])d\varphi\,\frac{\sin d\varphi}{\varphi}\exp(\psi(\varphi_t^{-1}[\varphi],t)-\beta\varphi) \qquad (32)$$

By the Riemann-Lebesgue lemma, we can evaluate $g(t)$;

$$g(t) = \frac{1}{2}\{f(\varphi_t^{-1}[0^+]) + f(\varphi_t^{-1}[0^-])\} = \frac{1}{2}\{f(t^+)+f(t^-)\} \qquad (33)$$

It follows from Green's formula that,

$$\int_0^\infty k(t,\lambda)\, L f(t)\, dt = \int_0^\infty L^* k(t,\lambda) f(t)\, dt - f(0) = \lambda \int_0^\infty k(t,\lambda) f(t)\, dt - f(0) \quad (34)$$

The formula for the transform of L^n follows by induction from the above.

VI. Generalized Fourier Transforms

In closing we point out that integral transforms of the type described in the first part of this material do not exhaust all avenues leading to an operational calculus. We refer now to the generalized Fourier transform of E. A. Coddington[6,7]. Coddington has shown that if L is an n^{th} order differential operator defined on $a < t < b$, (a,b) bounded or not, and if L is formally self adjoint, and if the coefficients of L have continuous derivatives of order as high as their respective indices, then the following is true: There exists an hermitian matrix of functions

$$[\rho_{ij}(\lambda)]$$

with elements of bounded variation on every finite λ-interval with this property. If $\{s_j(t,\lambda)\}$ is a set of n linearly independent solutions of $Ly = \lambda y$, then for a square summable function $f(t)$ it will be true that

$$f(t) = \int_{-\infty}^{\infty} \sum_{ij} s_i(t,\lambda) \left\{ \int_a^b s_j(\tau,\lambda) f(\tau)\, d\tau \right\} d\rho_{ij}(\lambda) \quad (35)$$

Furthermore, the Parseval equality takes the form,

$$\int_a^b |f(t)|^2 dt = \int_{-\infty}^{\infty} \sum_{i,j} \varphi_i(\lambda)\varphi_j(\lambda)\, d\rho_{ij}(\lambda) \tag{36}$$

where

$$\varphi_i(\lambda) = \int_a^b s_i(\tau,\lambda) f(\tau)\, d\tau$$

For our network applications we may take $a = 0$ and $b = \infty$. Coddington's transform, in effect maps each function $f(t)$ on an n-tuple of functions $\varphi_i(\lambda)$. Now the relation between the transform of voltage $[E_i(\lambda)]$ and the transform of current $[I_i(\lambda)]$ must be related by,

$$E_i(\lambda) = Z(\lambda)\, I_i(\lambda) \quad , \; i = 1, 2, \ldots, n \tag{37}$$

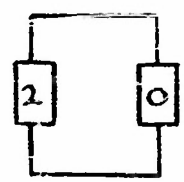
for any two terminal network of elements which are characterized by

$$\alpha^{(p)} L^p i(t) = e(t).$$

Thus, we again have a simple impedance function. Using the Parseval equality we can show to the same effect as before that a necessary and sufficient condition that $Z(\lambda)$ be the impedance function of a two terminal network of positive elements with $p = 1$, 0, and -1 only, is that $Z(\lambda)$ be real when λ is real and $\mathrm{Re}\, Z \geqq 0$ when $\mathrm{Re}\,\lambda \geqq 0$.

VII. Some Unanswered Questions

We conclude this report with a brief mention of some outstanding questions which remain for future study. We have seen how a class of problems in network analysis and synthesis can be handled if one possesses a functional transformation for which a form of Parseval's equality holds, and also which generates an operational calculus. If the restrictions to first order differential operators with only exponents 1, 0, and -1 is dropped, many new possibilities arise. For example, if $L = {}^{d}/dt$ and elements of type 2 are allowed, the circuit,



is not passive from an energy point of view, even though only positive elements are allowed. Thus, passivity is no longer synonymous with what is usually termed stability. A question is therefore raised: How can one differentiate between stability in the sense of bounded response, and instability in the sense of unbounded response?

An examination of the situation shows that if $Z(\lambda)$ is the impedance function of a two terminal network of positive elements, then $Z(\lambda)$ has the following property: $Z(\lambda)$ is real when λ is real; and further, if L is an n^{th} order differential operator, and if elements of type p, $-m_1 \leq p \leq m_2$ are present in the network, then the real part of $Z(\lambda)$ is positive when $-nm_1\pi/2 \leq \arg(\lambda) \leq nm_2\pi/2$

which is the shaded area below.

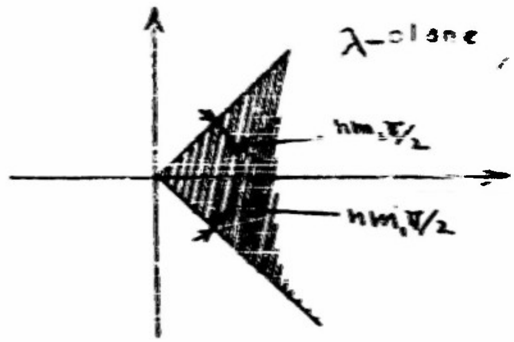


Figure 2

An interesting question to raise is whether, as is the case when $m_1 = m_2 = n = 1$, the converse is also true; i.e., if $Z(\ \ )$ has the above property whether or not there exists a network of positive elements of type p where $-m_1 \leq p \leq m_2$ whose impedance function is $Z(\lambda)$. Another question requiring an answer is to ascertain those conditions on $Z(\lambda)$ which insure stability of the corresponding network.

VIII. Review

Briefly, the point of view taken in this report is as follows: If one has a multiloop network of linear time-varying elements, and one knows the response of a typical branch to an arbitrary excitation; then formally, the response of the entire network is expressable as a linear combination of such typical responses (compare the operational calculus generated by the Laplace transform, and its application to fixed parameter networks). With the aid of certain integral transformations, the determination of the coefficients of the linear combination (which may be, in the limiting case an integral) is simplified to the formalism of an operational calculus. From an abstract point of view, this does not differ from the application of the Laplace transform to the study of fixed parameter networks. We have suggested how much of what might be called classical fixed parameter network theory has immediate extension to time-varying networks when such a formalism is at hand. For example, an elementary problem of time-varying network synthesis is presented which follows steps similar to fixed parameter network synthesis.

In addition to providing a discipline for the analysis and synthesis of time-varying networks, it provides a simple picture on which one can build a reliable intuition regarding certain time-varying networks. Although incomplete in several respects, this approach promises a better understanding of a class of problems of considerable interest.

REFERENCES

1. Aseltine, J. A., Transforms for Linear Time-Varying Systems, Tech. Report 52-1 Dept. of Engineering, University of California, Los Angeles.

2. Aseltine, J. A., and Trautman, D. L., Paper presented at the Institute of Radio Engineers Convention, March, 1952 New York, N. Y.

3. Davis, H., and Trautman, D. L., Paper presented URSI meeting, April, 1953, Washington, D. C.

4. Brune, Journal of Math. and Physics, Vol. 10 (1931) pp. 191-234, M.I.T.

5. Bott, R., and Duffin, R. J., Journal of Applied Physics, Vol. 20 (1949) p. 816.

6. Coddington, E. A., National Academy of Science, Vol. 38 (1952) pp. 732-737.

7. Coddington, E. A., Canadian Journal of Mathematics, Vol. 6 (1954) pp. 169-185.

8. Guillemin, E. A., Communications Networks, John Wiley and Sons, 1931.

9. Network Research Bulletins (University of California, Department of Engineering, Los Angeles; Departmental Memos, RP Nos. 5946, 6224, 6353, 8754, 9218, and 11349.

10. Davis, H., Ho, E. C., and Trautman, D. L., Extension of the Operational Calculus to Time-Varying Networks, presented June 23, 1954 at the Summer and Pacific General Meeting of the AIEE.

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