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MOTION EQUATIONS FOR TORPEDOES

By

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Underwater Ordnance Department

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FOREWORD

The science of underwater warfare has progressed rapidly in recent years, particularly in the development of torpedoes of greatly increased range and speed, and homing torpedoes which are required to run complicated trajectory patterns. This advance has been accompanied by the introduction of problems in the control of these new weapons. Intuitive, cut-and-try methods of design, which formerly were adequate for the control of the torpedo, are now too expensive and time-consuming. The test launching of a full-scale torpedo is an operation in which there is risk of damage, or loss of the missile. Since a torpedo is a costly weapon it is highly desirable that the behavior of the weapon be accurately predicted before it is ever launched. To this end it is necessary that the equations of motion and the mathematical expression of the laws of motion be well understood and be expressed in usable form.

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ABSTRACT

Motion equations are developed for a rigid torpedo of constant mass with six degrees of freedom. It is assumed that the medium is at rest except for the motion caused by the torpedo. The mathematical form of the "mass accession" forces is derived from potential theory. The motion equations are referred to body coordinates in their development, and transformations are made to inertial coordinates. Some sources of hydrodynamic coefficients are discussed, and an outline of the methods for obtaining them from model tests is presented. Solutions of steady-state equations are given, as well as a brief explanation of the analog computer method of solving the motion equations.

CONTENTS

INTRODUCTION

A great deal of research has been done in the past few years in the analysis of the motion of a torpedo. The problem of uncontrolled ship and torpedo motions was investigated by Schiff and Davidson (Ref. 1) and by Minorsky (Ref. 2 and 3). More recently Bednarz and Harger (Ref. 4) studied the problem with the purpose of giving a physical insight into the effects of the coefficients of the simplified motion equations. Triaxial motion equations were developed by Pierce and Sepmeyer (Ref. 5) for use in the Hydrodynamic Simulator. The standardization of nomenclature by the Committee on Nomenclature of the American Towing Tank Conference (1948) was an important step forward in the treatment of motion of a submerged body.

In the past the most serious obstacle to the analysis of torpedo motion has been the absence of information concerning the hydrodynamic forces acting on a torpedo. To eliminate this deficiency towing tanks and water tunnels were constructed. Data obtained in model tests at these tunnels yield the dimensionless coefficients that characterize the hydrodynamic behavior of a torpedo.

Techniques and facilities for the solution of the motion equations have been expanded, and it is now possible to solve complex control problems with relative ease. An important facility used in the analysis of torpedo motion is the electronic analog computer. With the aid of the computer linear or nonlinear equations are solved rapidly and accurately.

In this report the motion equations are developed on as firm a theoretical basis as present knowledge permits. An explanation of the methods of analysis may be found in Ref. 6. A discussion of the methods by which hydrodynamic coefficients are measured in model tests, and a very brief outline of the analog computer method of solution of the motion equations are included. Since an understanding of the theory of torpedo motion must precede applications to the design of new weapons, this report is presented with the hope that investigators in the fields of hydrodynamics and torpedo control will be aided in understanding the present state of the technique and be stimulated towards its advancement.

MOTION EQUATIONS

The reaction of a torpedo to external forces is expressed by the fundamental laws of dynamics. In the development of the motion equations in this report, it has been necessary to make assumptions about the nature of the torpedo. One assumption has been that it is a rigid body. The torpedo is elastic to some degree, of course, and has moving parts--properties which will probably be of interest in future studies. At present, however, it is felt that the assumption of rigidity is valid for the applications that the equations of this report have in view. It is assumed, moreover, that the torpedo is of constant mass. This is a better assumption for electric torpedoes than for turbine or engine driven torpedoes, since a considerable quantity of fuel is consumed in the latter. Usually the rate of fuel consumption is so slow that it has little effect on the trajectory. In a particular problem, however, an investigation should be made as to the length of the trajectory for which changes in the magnitude of the inertia of the torpedo may be neglected. In particular problems, moreover, it may be necessary to modify the equations as they are given here. For example, it has been assumed that the thrust of the propulsion system acts along the longitudinal axis of the torpedo without resultant torque. In some cases it may be necessary to add additional terms to the equations if the thrust is misaligned or if an unbalanced torque is present. It is assumed the torpedo is fully wetted. If it is in a cavitating state the equations given here remain applicable, but a modification of the hydrodynamic forces is necessary because they are then functionally related to the cavitation parameter.

BASIC VECTORIAL EQUATIONS OF MOTION

The laws of motion are applied to a torpedo under the assumption that it is a rigid body. The basic equations are

$$(1) \qquad \underline{F} = \frac{d\underline{G}_b}{dt}$$

$$(2) \qquad \underline{L} = \frac{d\underline{H}_b}{dt}$$

where $\underline{F}$ is the resultant external force applied to the torpedo body, $\underline{G}_b$ is its linear momentum, $\underline{L}$ is the resultant moment acting on the torpedo, and $\underline{H}_b$ is its total angular momentum. The time rate of change of the torpedo momenta $\underline{G}_b$ and $\underline{H}_b$ must be with respect to an inertial reference frame for the application of the dynamic equations. A right-handed rectangular reference frame (x_o, y_o, z_o) fixed with respect to the earth will be used for this purpose. The x_o-y_o plane is tangent to the surface of the earth, and the z_o axis is vertically downward. Use of this reference frame as an inertial reference frame implies the assumption that the motion of the earth has a negligible effect on the trajectory of the torpedo.

An underwater missile in accelerated motion produces accelerations in the flow of the fluid in which it is moving. Consequently there is a transfer of kinetic energy to the fluid. The rate of change of this kinetic energy, and therefore the force producing it, is proportional to the acceleration of the missile. Since the inertial reaction of the missile is also proportional to acceleration, the missile behaves as if its mass were increased. This phenomenon is termed "mass accession". It will be assumed in this report that the mathematical form of the "mass accession" forces is given by the theory of ideal fluid flow.

Let $\underline{F}_1$ and $\underline{L}_1$ be respectively the force and the moment (predicted by an ideal fluid) on the torpedo, and let

$$\underline{F}_2 = \underline{F} - \underline{F}_1 \tag{3}$$

$$\underline{L}_2 = \underline{L} - \underline{L}_1 \tag{4}$$

The force and moment on the torpedo predicted from potential flow are equal, respectively, to the negative of the time rates of change of the linear momentum and the angular momentum of the fluid (see Appendix B). Thus

$$\underline{F}_1 = -\frac{d\underline{G}_f}{dt} \tag{5}$$

$$\underline{L}_1 = -\frac{d\underline{H}_f}{dt} \tag{6}$$

By defining the system of body and ideal fluid momenta as

$$\underline{G} = \underline{G}_f + \underline{G}_b \tag{7}$$

$$\underline{H} = \underline{H}_f + \underline{H}_b \tag{8}$$

the basic motion equations may be written

$$\underline{F}_2 = \frac{d\underline{G}}{dt} \tag{9}$$

$$\underline{L}_2 = \frac{d\underline{H}}{dt} \tag{10}$$

It is advantageous to employ a frame of reference fixed with respect to the torpedo body. The reference frame moves with the velocity $\underline{V}$ of the torpedo and rotates with the angular velocity $\underline{\omega}$. In this reference system (see Appendix C)

$$\underline{F}_2 = \mathring{\underline{G}} + \underline{\omega} \times \underline{G} \tag{11}$$

$$\underline{L}_2 = \mathring{\underline{H}} + \underline{\omega} \times \underline{H} + \underline{V} \times \underline{G} \tag{12}$$

where $\mathring{\underline{G}}$ and $\mathring{\underline{H}}$ represent the time rates of change of $\underline{G}$ and $\underline{H}$ as seen from the moving system.

The momenta of the system may be evaluated from the total kinetic energy T of the system. Let T be expressed as a function of the components of the velocities $\underline{V}$ and $\underline{\omega}$ in the body reference frame. Then, letting

$$\underline{V} = \underline{i}U + \underline{j}v + \underline{k}w \tag{13}$$

and

$$\underline{\omega} = \underline{i}p + \underline{j}q + \underline{k}r \tag{14}$$

the momenta are given by

$$\underline{G} = \underline{i}\frac{\partial T}{\partial U} + \underline{j}\frac{\partial T}{\partial v} + \underline{k}\frac{\partial T}{\partial w} \tag{15}$$

$$\underline{H} = \underline{i}\frac{\partial T}{\partial p} + \underline{j}\frac{\partial T}{\partial q} + \underline{k}\frac{\partial T}{\partial r} \tag{16}$$

(see Appendix D). The total kinetic energy T is the sum of the kinetic energy T_b of the fluid and the kinetic energy T_f of the torpedo body. These are derived in the following section.

KINETIC ENERGY OF SYSTEM

Kinetic Energy of the Torpedo Body

The torpedo body will be considered a rigid aggregate of mass particles. Let m_i be a representative particle, and let $\underline{r_i}$ be the radius vector from the origin of body coordinates to m_i. The velocity of m_i is

$$\underline{v_i} = V + \underline{\omega} \times \underline{r_i} \tag{17}$$

The kinetic energy of the body is equal to the sum of the kinetic energies of the individual particles of mass, and is given by

$$T_b = (1/2) \sum m_i \left|\underline{v_i}\right|^2 \tag{18}$$

Let $\underline{r_i}$ have the body coordinates x_i, y_i, z_i. The expansion of Eq. 18 yields:

$$\begin{aligned} 2T_b = \sum m_i \Big[& U^2 + v^2 + w^2 + (y_i^2 + z_i^2)p^2 + (x_i^2 + z_i^2)q^2 \\ & + (x_i^2 + y_i^2)\, r^2 + 2Uqz_i - 2Ury_i - 2qry_iz_i + 2vrx_i \\ & - 2vpz_i - 2rpx_iz_i + 2wpy_i - 2wqx_i - 2pqx_iy_i \Big] \end{aligned} \tag{19}$$

The following quantities are defined:

$$\begin{array}{llcllcllc} \sum m_i = m, \text{ torpedo mass} & & & & & & & & \\ \sum m_i(y_i^2 + z_i^2) = I_x & & & \sum m_iy_iz_i = I_{yz} & & & \sum m_ix_i = mx_G & & \\ \sum m_i(x_i^2 + z_i^2) = I_y & & \text{mom. of inertia} & \sum m_ix_iz_i = I_{xz} & & \text{prod. of inertia} & \sum m_iy_i = my_G & & \text{C.G. coordinates} \\ \sum m_i(x_i^2 + y_i^2) = I_z & & & \sum m_ix_iy_i = I_{xy} & & & \sum m_iz_i = mz_G & & \end{array} \tag{20}$$

Equation 19 then becomes

$$\begin{aligned} 2T_b = {} & mU^2 + mv^2 + mw^2 + I_xp^2 + I_yq^2 + I_zr^2 \\ & + 2mz_GUq - 2my_GUr - 2I_{yz}qr + 2mx_Gvr \\ & - 2mz_Gvp - 2I_{xz}rp + 2my_Gwp - 2mx_Gwq - 2I_{xy}pq \end{aligned} \tag{21}$$

The origin of the body coordinate system is usually placed on the longitudinal axis of the torpedo above the center of gravity. The positive x-axis is in the forward direction of the longitudinal axis, and the positive z-axis is vertically downward through the center of gravity. The choice of this reference frame results in the simplification of the expression for T_b, since in this case

$$x_G = y_G = I_{xy} = I_{yz} = 0$$

Kinetic Energy of Ideal Fluid

Suppose the torpedo to be moving in the direction of its longitudinal axis with the velocity $U = U(t)$. It will be assumed that the flow produced is irrotational and that the fluid is non-viscous and incompressible. Let the velocity field of the flow be $\underline{q}(x,y,z,t)$. Because the flow is irrotational a potential function $\phi(x,y,z,t)$ exists such that

(22) $$\underline{q} = -\nabla\phi$$

Since the fluid is incompressible, the divergence of the velocity vanishes; and consequently

(23) $$\operatorname{div} \underline{q} = -\nabla^2\phi = 0$$

There can be no flow across the surface of the torpedo. Hence the normal component of a point on the surface of the torpedo must equal the normal component of the fluid velocity at that point. Let the unit normal to the surface, drawn toward the fluid, be

(24) $$\underline{n} = \underline{i}\lambda + \underline{j}\mu + \underline{k}\nu$$

On the surface, then,

(25) $$-\frac{\partial\phi}{\partial n} = \lambda U$$

It is assumed that the flow is started from rest, and it is impossible that finite forces acting for a finite time produce a flow with infinite kinetic energy. Therefore, the velocity of the flow must vanish at an infinite distance from the torpedo, since a finite velocity at an infinite distance would imply an infinite kinetic energy. Hence a potential function ϕ is required which satisfies Eq. 25 on the surface of the torpedo and whose gradient vanishes at infinity. A solution is sought having the form

(26) $$\phi = U(t)\phi_1\ (x,y,z)$$

Since $\nabla^2\phi_1 = 0$ with $\partial\phi_1/\partial n = -\lambda$ on the surface of the torpedo and $\nabla\phi_1 = 0$ at infinity, the function ϕ_1 is uniquely determined. Hence Eq. 26 is the solution to the flow problem, since an irrotational flow with vanishing divergence is uniquely determined by its boundary conditions.

Suppose now that the torpedo is rotating about its longitudinal axis with an angular velocity $p = p(t)$. As before, a velocity potential exists satisfying Eq. 23. Moreover, at a point (x, y, z) of the surface of the torpedo

$$- \frac{\partial \phi}{\partial n} = p(\nu y - \mu z) \tag{27}$$

A solution is sought of the form

$$\phi = p(t)\phi_4(x, y, z) \tag{28}$$

Since $\nabla^2\phi_4 = 0$, $- \dfrac{\partial \phi_4}{\partial n} = \nu y - \mu z$ on the surface of the torpedo, and $\nabla_4 = 0$ at infinity, ϕ_4 is uniquely determined. Hence Eq. 28 is the solution to this flow problem. Several velocity distributions $\underline{q}_1$, $\underline{q}_2$, $\underline{q}_3$, . . . may be added together to obtain another velocity distribution. The velocity distribution

$$\underline{q} = \underline{q}_1 + \underline{q}_2 + \underline{q}_3 + \cdots$$

is said to be the superposition, or rather the result of the superposition, of the velocity distributions $\underline{q}_1$, $\underline{q}_2$, $\underline{q}_3$, ... It is evident that if the divergence or the rotation of each of the velocity distributions vanishes, the divergence or rotation of their superposition vanishes also.

Now consider the flows produced by motion of the torpedo in each of its remaining degrees of freedom. For motion in the direction of the y-axis a potential $v\phi_2$, and for motion in the direction of the z-axis a potential $w\phi_3$ is obtained. Rotary motion about the y-axis yields a potential $q\phi_5$, and motion about the z-axis yields $r\phi_6$. The potential function for the flow produced by motion of the torpedo in its six degrees of freedom is obtained, using the principle of superposition, as

$$\phi = U\phi_1 + v\phi_2 + w\phi_3 + p\phi_4 + q\phi_5 + r\phi_6 \tag{29}$$

Let T_f be the kinetic energy of the fluid. Then

$$T_f = (1/2)\rho\int_{\tau} (\nabla\phi)^2 \, d\tau \tag{30}$$

where the integration is over the entire volume τ of the fluid. The integral may be transformed by Green's theorem to

(31)
$$T_f = -(1/2)\rho\int_S \phi \frac{\partial\phi}{\partial n} dS \; -(1/2)\rho\int_\tau \phi\nabla^2\phi \; d\tau$$

$$= -(1/2)\rho\int_S \phi \frac{\partial\phi}{\partial n} dS \text{ since } \nabla^2\phi = 0$$

where the integration is over the torpedo surfaces. Substituting Eq. 29 into Eq. 31 yields a quadratic form in the torpedo velocity components,

(32)
$$\begin{aligned} 2T_f = {} & a_{11}U^2 + a_{12}Uv + a_{13}Uw + a_{14}Up + a_{15}Uq + a_{16}Ur \\ & + a_{21}vU + a_{22}v^2 + a_{23}vw + a_{24}vp + a_{25}vq + a_{26}vr \\ & + a_{31}wU + a_{32}wv + a_{33}w^2 + a_{34}wp + a_{35}wq + a_{36}wr \\ & + a_{41}pU + a_{42}pv + a_{43}pw + a_{44}p^2 + a_{45}pq + a_{46}pr \\ & + a_{51}qU + a_{52}qv + a_{53}qw + a_{54}qp + a_{55}q^2 + a_{56}qr \\ & + a_{61}rU + a_{62}rv + a_{63}rw + a_{64}rp + a_{65}rq + a_{66}r^2 \end{aligned}$$

where

(33)
$$a_{ij} = -\rho\int_S \phi_i \frac{\partial\phi_j}{\partial n} dS$$

It will be noted that Green's theorem gives

(34)
$$a_{ij} = a_{ji}$$

Suppose the surface of the torpedo to be symmetric with respect to the x-z plane and with respect to the x-y plane. For translatory motion in the x-y plane,

(35)
$$2T_f = a_{11}U^2 + 2a_{12}Uv + a_{22}v^2$$

Because of the symmetry of the torpedo the kinetic energy must be unchanged if v is replaced by -v. Hence $a_{12} = 0$. It may be similarly shown that all the coefficients of cross-product terms vanish except a_{26}, a_{62}, a_{53}, and a_{35}. Equation 32 then reduces to

(36) $$2T_f = a_{11}U^2 + a_{22}v^2 + a_{33}w^2 + a_{44}p^2 + a_{55}q^2 + a_{66}r^2 + 2a_{35}wq + 2a_{26}vr$$

The linear momentum and the angular momentum of the fluid may be obtained from T_f as is shown in Appendix D.

REACTION OF TORPEDO AND IDEAL FLUID SYSTEM

The total kinetic energy of the system composed of torpedo and ideal fluid is $T = T_b + T_f$

(37) $$2T = (m + a_{11})U^2 + (m + a_{22})v^2 + (m + a_{33})w^2 + (I_x + a_{44})p^2 + (I_y + a_{55})q^2 + (I_z + a_{66})r^2 + 2mz_GUq - 2I_{xz}rp - 2mz_Gvp + 2a_{35}wq + 2a_{26}vr$$

Define

(38)
$$m + a_{11} = m_L$$
$$m + a_{22} = m + a_{33} = m_T$$
$$I_x + a_{44} = J_x$$
$$I_y + a_{55} = J_y$$
$$I_z + a_{66} = J_z$$

Then Eq. 37 becomes

(39) $$2T = m_LU^2 + m_Tv^2 + m_Tw^2 + J_xp^2 + J_yq^2 + J_zr^2 + 2mz_GUq - 2I_{xz}rp - 2mz_Gvp + 2a_{35}wq + 2a_{26}vr$$

The components of momenta defined in Eq. 15 and Eq. 16 are

(40)
$$G_x = m_LU + mz_Gq$$
$$G_y = m_Tv - mz_Gp + a_{26}r$$
$$G_z = m_Tw + a_{35}q$$

(40) Continued $H_x = J_x p - I_{xz} r - mz_G v$

$$H_y = J_y q + mz_G U + a_{35} w$$

$$H_z = J_z r - I_{xz} p + a_{26} v$$

Let $\underline{F_2}$ have the components X_2, Y_2, Z_2, and $\underline{L_2}$ the components K_2, M_2, N_2. The basic motion equations (Eq. 11 and 12) written in terms of their components become

(41)
$$X_2 = \dot{G}_x + qG_z - rG_y$$

$$Y_2 = \dot{G}_y + rG_x - pG_z$$

$$Z_2 = \dot{G}_z + pG_y - qG_x$$

$$K_2 = \dot{H}_x + qH_z - rH_y + vG_z - wG_y$$

$$M_2 = \dot{H}_y + rH_x - pH_z + wG_x - UG_z$$

$$N_2 = \dot{H}_z + pH_y - qH_x + UG_y - vG_x$$

Substitution of Eq. 40 into Eq. 41 gives

(42)
$$X_2 = m_L\dot{U} + mz_G\dot{q} + m_T(wq - vr) + a_{35}q^2 + mz_G pr - a_{26}r^2$$

$$Y_2 = m_T\dot{v} - mz_G\dot{p} + a_{26}\dot{r} + m_L Ur + mz_G qr - m_T wp - a_{35}pq$$

$$Z_2 = m_T\dot{w} + a_{35}\dot{q} + m_T vp - mz_G(p^2 + q^2) + a_{26}pr - m_L Uq$$

$$K_2 = J_x\dot{p} - I_{xz}\dot{r} - mz_G\dot{v} + (J_z - J_y)qr - I_{xz}pq + (a_{26} + a_{35})vq$$
$$- mz_G Ur - (a_{35} + a_{26})\,wr + mz_G wp$$

$$M_2 = J_y\dot{q} + mz_G\dot{U} + a_{35}\dot{w} + (J_x - J_z)pr + I_{xz}(p^2 - r^2) - mz_G vr$$
$$- a_{26}vp + (m_L - m_T)U + mz_G wq - a_{35}Uq$$

$$N_2 = J_z\dot{r} - I_{xz}\dot{p} + a_{26}\dot{v} + (J_y - J_x)pq + a_{35}wp + I_{xz}qr$$
$$+ (m_T - m_L)Uv + a_{26}Ur$$

These, then, are the basic motion equations for a torpedo referred to a set of axes fixed with respect to the torpedo, the positive x-axis in the forward direction of the longitudinal axis, and the z-axis vertically downward through the center of gravity of the torpedo. Assumptions under which they have been derived are

1. The torpedo is a rigid body of constant mass, symmetric with respect to a plane through its longitudinal axis.

2. Motion of the earth has negligible effect on the trajectory of the torpedo.

3. The medium is infinite in extent and at rest except for the flow produced by the motion of the torpedo.

4. The torpedo is fully wetted.

5. Mass accession forces are formally given by the theory of ideal fluid flow.

EXTERNAL FORCES AND MOMENTS ACTING ON THE TORPEDO

The external forces and moments acting on the torpedo are those caused by gravity and the propulsion system, and those produced by hydrodynamic and hydrostatic pressures. The net force produced by gravity is the weight of the torpedo acting vertically downward at the center of gravity. The net force of the hydrostatic pressures is a buoyant force acting at the center of buoyancy of the torpedo. Resolution of these forces and the moments produced by them onto body coordinates is given in Appendix E. The thrust of the propulsion system will be assumed to act along the longitudinal axis of the torpedo with no resultant torque.

Hydrodynamic forces predicted by the theory of ideal fluid were discussed above. This theory predicts no lift or drag on the torpedo. Since a torpedo does, in fact, experience lift and drag forces, they are ascribed to deviation of the fluid motion from potential flow because of the viscosity of the fluid. At very high Reynolds numbers this deviation from potential flow will take place in a thin layer in the negihborhood of the surface of the torpedo (see Ref. 7). Particles of fluid at the surface of the torpedo adhere firmly to it so that not only the normal component of the fluid velocity at the surface but also the tangential velocity is equal to that of the surface. Outside the boundary layer a potential flow will exist. Let it be assumed that this potential is the same as Eq. 29.

The Navier-Stokes equations for viscous fluid flow give the force per unit volume acting on the fluid as

$$\underline{f} = -\nabla P + k\nabla^2 \underline{q} \tag{43}$$

where P is the pressure, $\underline{q}$ is the velocity field, and k is the coefficient of viscosity of the fluid. Integrating over the entire volume of the fluid gives the total force acting on the fluid as

$$\int_\tau \underline{f}\, d\tau = -\int_\tau \nabla P\, d\tau + k\int_\tau \nabla^2 \underline{q}\, d\tau = \int_S P\underline{n}\, dS - k\int_S \frac{\partial \underline{q}}{\partial n} dS \tag{44}$$

where the surface integrals are over the surface of the torpedo, and $\underline{n}$ is the unit normal to the surface projecting into the fluid. It is assumed that at the upper limit of the boundary layer $\underline{q} = -\nabla \phi$. At a point of the surface whose radius vector from the origin of body coordinates is $\underline{r}$, the velocity is $\underline{V} + \underline{\omega} \times \underline{r}$. Let the thickness of the boundary layer be denoted by δ. Then, approximately,

$$\frac{\partial \underline{q}}{\partial n} = -(\nabla\phi + \underline{V} + \underline{\omega} \times \underline{r})\delta^{-1} \tag{45}$$

The difference in pressure between the inner and outer surfaces of the boundary layer is small (Ref. 7). Hence, the additional force on the torpedo because of the viscosity of the fluid is approximately

$$-\int_S (\nabla\phi + \underline{V} + \underline{\omega} \times \underline{r})k\delta^{-1}\, dS \tag{46}$$

Since

$$(\underline{V} + \underline{\omega} \times \underline{r})\cdot\underline{n} = -(\nabla\phi)\cdot\underline{n} \text{ at the surface,}$$

$-\nabla\phi - (\underline{V} + \underline{\omega} \times \underline{r})$ represents the relative velocity at which the potential flow is sliding over the surface. This is a linear function of the torpedo velocity components. If δ were independent of the velocity components, then, the viscosity-induced force would be also a linear function of the torpedo-velocity components. This, however, is not the case. The boundary-layer thickness is a function of Reynolds number and the form of the surface. Moreover, the boundary layer may become quite thick and separate toward the after end of the torpedo, and the analysis given above is then not applicable. It is at least reasonable to assume, however, that the viscosity forces and moments are functions only of the torpedo velocity components.

The remaining hydrodynamic forces to be taken into consideration are those produced by deflections of the control surfaces.

These forces depend primarily on the magnitude of the deflection of the control surface and not on the rate in control systems used at present. It will be assumed, then, that the forces and moments produced by control surface deflections are functions of the magnitudes of the deflections, δ_e and δ_r.

The forces and moments of Eq. 11 and 12 may therefore be written

$$\underline{F_2} = \overline{\underline{F_2}}(U, v, w, p, q, r, \delta_e, \delta_r) + \underline{B} + \underline{W} + \underline{T} \tag{47}$$

$$\underline{L_2} = \overline{\underline{L_2}}(U, v, w, p, q, r, \delta_e, \delta_r) + \underline{r_B} \times \underline{B} + \underline{r_G} \times \underline{W}$$

where $\underline{B}$ is the buoyant force, $\underline{W}$ is the weight of the torpedo, $\underline{T}$ is the thrust, $\underline{r_B}$ is the radius vector to the center of buoyancy, and $\underline{r_G}$ is the radius vector to the center of gravity.

Let the components of $\overline{\underline{F_2}}$ be $\overline{X}_2$, $\overline{Y}_2$, $\overline{Z}_2$ and the components of $\overline{\underline{L}}$ be $\overline{K}_2$, $\overline{M}_2$, $\overline{N}_2$. These forces and moments are usually determined in model studies (see section entitled "Sources of Hydrodynamic Coefficients"). Results obtained from these studies show that, for most torpedoes, the hydrodynamic forces and moments are approximately linear functions of angle of attack and turning rate over the normal operating range. Forces and moments produced by control-surface deflections are also linear over a wide range for most torpedoes. There is some justification, therefore, for expanding the components of $\underline{F_2}$ and $\underline{L_2}$ in a Taylor series about $U = U_o$, $v = w = p = q = r = \delta_e = \delta_r = 0$, where U_o is the operating velocity of the torpedo, and neglecting all but first order terms. Hence, letting X_{20} be $\overline{X}_2$ evaluated at the point about which the series expansion is made, and similarly with the other components, and letting $U = U_o + u$,

$$\begin{aligned} \overline{X}_2 &= X_{20} + X_{2u}u + X_{2v}v + X_{2w}w + X_{2p}p + X_{2q}q \\ &\quad + X_{2r}r + X_{2\delta_e}\delta_e + X_{2\delta_r}\delta_r \end{aligned} \tag{48}$$

$$\begin{aligned} \overline{Y}_2 &= Y_{20} + Y_{2u}u + Y_{2v}v + Y_{2w}w + Y_{2p}p + Y_{2q}q \\ &\quad + Y_{2r}r + Y_{2\delta_e}\delta_e + Y_{2\delta_r}\delta_r \end{aligned}$$

$$\begin{aligned} \overline{Z}_2 &= Z_{20} + Z_{2u}u + Z_{2v}v + Z_{2w}w + Z_{2p}p + Z_{2q}q \\ &\quad Z_{2r}r + Z_{2\delta_e}\delta_e + Z_{2\delta_r}\delta_r \end{aligned}$$

(48) Contd. $\bar{K}_2 = K_{20} + K_{2u}u + K_{2v}v + K_{2w}w + K_{2p}p + K_{2q}q$

$$+ K_{2r}r + K_{2\delta_e}\delta_e + K_{2\delta_r}\delta_r$$

$$\bar{M}_2 = M_{20} + M_{2u}u + M_{2v}v + M_{2w}w + M_{2p}p + M_{2q}q$$

$$+ M_{2r}r + M_{2\delta_e}\delta_e + M_{2\delta_r}\delta_r$$

$$\bar{N}_2 = N_{20} + N_{2u}u + N_{2v}v + N_{2w}w + N_{2p}p + N_{2q}q$$

$$+ N_{2r}r + N_{2\delta_e}\delta_e + N_{2\delta_r}\delta_r$$

where the partial derivatives are evaluated at

$$u = v = w = p = q = r = \delta_e = \delta_r = 0$$

Because of symmetry with respect to the x-z plane, and because of the point about which the Taylor expansion is made, the following partial derivatives and component values at the point of expansion vanish:

$$X_{2v} = X_{2p} = X_{2r} = X_{2q} = X_{2w} = X_{2\delta e} = X_{2\delta r} = 0$$

$$Y_{2u} = Y_{2w} = Y_{2q} = Y_{2\delta e} = Y_{20} = 0$$

$$Z_{2u} = Z_{2v} = Z_{2p} = Z_{2r} = Z_{2\delta r} = Z_{20} = 0$$

$$K_{2u} = K_{2w} = K_{2q} = K_{2\delta e} = K_{20} = 0$$

$$M_{2u} = M_{2v} = M_{2p} = M_{2r} = M_{2\delta r} = M_{20} = 0$$

$$N_{2u} = N_{2w} = N_{2q} = N_{2\delta e} = N_{20} = 0$$

Using the resolution of buoyancy and gravity forces and moments as given in Appendix E and the linearized external forces (Eq. 48) the equations of motion (Eq. 42) become

(49) $X_{20} + X_{2u}u + \underline{T} - (W - B)\sin\theta = m_L\dot{u} + mz_G\dot{q} + m_T(wq - vr)$

$$+ a_{35}q^2 + mz_G pr - a_{26}r^2$$

(49) Contd.

$$Y_{2v}v + Y_{2p}p + Y_{2r}r + Y_{2\delta_r}\delta_r + (W - B)\sin\phi\cos\theta$$

$$= m_T\dot{v} - mz_G\dot{p} + a_{26}\dot{r} + m_L Ur + mz_G qr - m_T wp - a_{35}pq$$

$$Z_{2w}w + Z_{2q}q + Z_{2\delta_e}\delta_e + (W - B)\cos\phi\cos\theta$$

$$= m_T\dot{w} + a_{35}\dot{q} + m_T vp - mz_G(p^2 + q^2) + a_{26}pr - m_L Uq$$

$$K_{2v}v + K_{2p}p + K_{2r}r + K_{2\delta_r}\delta_r - Wz_G\sin\phi\cos\theta$$

$$= J_x\dot{p} - I_{xz}\dot{r} - mz_G\dot{v} + (J_z - J_y)\,qr - I_{xz}pq + (a_{26} + a_{35})vq$$

$$- mz_G Ur - (a_{35} + a_{26})wr + mz_G wp$$

$$M_{2w}w + M_{2q}q + M_{2\delta_e}\delta_e + Bx_B\cos\phi\cos\theta - Wz_G\sin\theta$$

$$= J_y\dot{q} + mz_G\dot{u} + a_{35}\dot{w} + (J_x - J_z)pr + I_{xz}(p^2 - r^2)$$

$$- mz_G vr - a_{26}vp + (m_L - m_T)Uw + mz_G wq - a_{35}Uq$$

$$N_{2v}v + N_{2p}p + N_{2r}r + N_{2\delta_r}\delta_r - Bx_B\sin\phi\cos\theta$$

$$= J_z\dot{r} - I_{xz}\dot{p} + a_{26}\dot{v} + (J_y - J_x)pq + a_{35}wp + I_{xz}qr$$

$$+ (m_T - m_L)Uv + a_{26}Ur$$

The motion equations (Eq. 49) will be rewritten in terms of angle of attack α and angle of sideslip β. By definition

(50) $$\sin\alpha = \frac{w}{V}$$

$$\sin\beta = -\frac{v}{V}$$

It will be assumed that α and β are small angles so that the sine of the angle is approximated by the angle. Moreover, a change of notation will be introduced at this point. Let

$$X_o = X_{2o}$$

$$X_u = X_{2u}$$

$$K_{\delta_e} = K_{2\delta_e}$$

$$K_{\delta_r} = K_{2\delta_r}$$

$$K_\beta = -K_{2v}V$$

$$K_p = K_{2p}$$

$$K_r = K_{2r}$$

$$N_\beta = -N_{2v}V + (m_T - m_L)V^2$$

$$N_p = N_{2p}$$

$$N_r = N_{2r} - a_{26}V$$

$$N_{\dot\beta} = a_{26}V$$

$$N_{\delta_r} = N_{2\delta_r}$$

$$Y_\beta = -Y_{2v}V$$

$$Y_p = Y_{2p}$$

$$Y_r = Y_{2r}$$

$$Y_{\dot r} = -a_{26}$$

$$Y_{\delta_r} = Y_{2\delta_r}$$

$$M_\alpha = M_{2w}V + (m_T - m_L)V^2$$

$$M_q = M_{2q} + a_{35}V$$

$$M_{\dot\alpha} = -a_{35}V$$

$$M_{\delta_e} = M_{2\delta_e}$$

$$Z_\alpha = Z_{2w}V$$

$$Z_q = Z_{2q}$$

$$Z_{\dot q} = -a_{35}$$

$$Z_{\delta_e} = Z_{2\delta_e}$$

With this change of notation the equations of motion (Eq. 49) become

(51)
$$X_o + X_u u + \underline{T} - (W - B)\sin\theta$$
$$= m_L\dot u + mz_G\dot q + m_T V(\alpha q + \beta r) + mz_G pr - Z_{\dot q}q^2 + V_{\dot r}r^2$$

$$Y_\beta\beta + Y_p p + Y_r r + Y_{\dot r}\dot r + Y_{\delta_r}\delta_r + (W - B\cos\theta\sin\varphi$$
$$= -m_T V\dot\beta - mz_G\dot p + m_L Ur + mz_G qr + Z_{\dot q}pq - m_T V\alpha p$$

$$Z_\alpha\alpha + Z_q q + Z_{\dot q}\dot q + Z_{\delta_e}\delta_e + (W - B)\cos\phi\cos\theta$$
$$= m_T V\dot\alpha - m_T V\beta p - mz_G(p^2 + q^2) - Y_{\dot r}pr - m_L Uq$$

(51) Contd.

$$K_\beta\beta + K_p p + K_r r + K_{\delta_r}\delta_r - Wz_G \sin\phi\cos\theta$$

$$= J_x\dot{p} - I_{xz}\dot{r} + mz_G V\dot{\beta} + (J_z - J_y)qr - I_{xz}pq$$

$$+ V(Y_{\dot{r}} + Z_{\dot{q}})(\beta q + \alpha r) + mz_G Ur + mz_G V\alpha p$$

$$M_\alpha\alpha + M_q q + M_{\dot{\alpha}}\dot{\alpha} + M_{\delta_e}\delta_e + Bx_B\cos\phi\cos\theta - Wz_G\sin\theta$$

$$= J_y\dot{q} + mz_G\dot{u} + (J_x - J_z)pr + I_{xz}(p^2 - r^2)$$

$$+ mz_G V\beta r + N_{\dot{\beta}}\beta p + mz_G V\alpha q$$

$$N_\beta\beta + N_p p + N_r r + N_{\dot{\beta}}\dot{\beta} + N_{\delta_r}\delta_r - Bx_B\sin\phi\cos\theta$$

$$= J_z\dot{r} - I_{xz}\dot{p} + (J_y - J_x)pq - M_{\dot{\alpha}}\alpha p + I_{xz}qr$$

TRANSFORMATIONS OF MOTION EQUATIONS

The motion equations may be written in terms of the inertial angular position of the torpedo, ψ, θ, φ, (Ref. 4). The components of the rotational velocity of the torpedo, p, q, and r, in body coordinates are related to ψ, θ, ϕ by

(52)
$$p = \dot{\phi} - \dot{\psi}\sin\theta$$
$$q = \dot{\psi}\cos\theta\sin\varphi + \dot{\theta}\cos\varphi$$
$$r = \dot{\psi}\cos\theta\cos\varphi - \dot{\theta}\sin\varphi$$

(see Appendix F). Components of the angular acceleration, $\dot{p}$, $\dot{q}$, and $\dot{r}$, are subsequently found to be

(53)
$$\dot{p} = \ddot{\phi} - \ddot{\psi}\sin\theta - \dot{\theta}\dot{\psi}\cos\theta$$
$$\dot{q} = \ddot{\psi}\cos\theta\sin\varphi - \dot{\psi}\dot{\theta}\sin\theta\sin\varphi + \dot{\psi}\dot{\varphi}\cos\varphi\cos\theta$$
$$+ \ddot{\theta}\cos\varphi - \dot{\theta}\dot{\varphi}\sin\varphi$$
$$\dot{r} = \ddot{\psi}\cos\theta\cos\varphi - \dot{\psi}\dot{\theta}\sin\theta\cos\varphi - \dot{\psi}\dot{\varphi}\cos\theta\sin\varphi$$
$$- \ddot{\theta}\sin\varphi - \dot{\theta}\dot{\varphi}\cos\varphi$$

In addition to the assumptions already made it will be assumed that the pitch angle θ is small so that approximately, $\sin\theta = \theta$, $\cos\theta = 1$. Using Eq. 52 and 53 the motion equations (Eq. 51) are expressed in terms of inertial angles as

$$
\begin{aligned}
(54)\quad & X_o + X_u u + T_x - (W - B)\theta \\
& \quad = m_L \dot{u} + m z_G(\ddot{\psi}\sin\varphi - \dot{\psi}\dot{\theta}\theta \sin\varphi + \dot{\psi}\dot{\varphi}\cos\varphi + \ddot{\theta}\cos\varphi \\
& \qquad - \dot{\theta}\dot{\varphi}\sin\varphi) + m_T V\Big[\alpha(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi) \\
& \qquad + \beta(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi)\Big] - Z_q(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi)^2 \\
& \qquad + Y_r(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi)^2 + m z_G(\dot{\varphi} - \dot{\psi}\theta)(\dot{\psi}\cos\varphi - \theta\sin\varphi) \\
& Y_\beta\beta + Y_p(\dot{\varphi} - \dot{\psi}\theta) + Y_r(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi) + Y_{\dot{r}}(\ddot{\psi}\cos\varphi \\
& \qquad - \dot{\psi}\dot{\theta}\theta\cos\varphi - \dot{\psi}\dot{\varphi}\sin\varphi - \ddot{\theta}\sin\varphi - \dot{\theta}\dot{\varphi}\cos\varphi) \\
& \qquad + Y_{\delta_r}\delta_r + (W - B)\sin\varphi = -m_T V\dot{\beta} - m z_G(\ddot{\varphi} - \ddot{\psi}\theta - \dot{\theta}\dot{\psi}) \\
& \qquad + m_L U(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi) \\
& \qquad + m z_G(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi)(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi) \\
& \qquad - m_T V\alpha(\dot{\varphi} - \dot{\psi}\theta) + Z_{\dot{q}}(\dot{\varphi}' - \dot{\psi}\theta)(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi) \\
& Z_\alpha\alpha + Z_q(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi) + Z_{\dot{q}}(\ddot{\psi}\sin\varphi - \dot{\psi}\dot{\theta}\theta\sin\varphi \\
& \qquad + \dot{\psi}\dot{\varphi}\cos\varphi + \ddot{\theta}\cos\varphi - \dot{\theta}\dot{\varphi}\sin\varphi) + Z_{\delta_e}\delta_e + (W - B)\cos\varphi \\
& \qquad\quad = m_T V\dot{\alpha} - m_T V\beta\,(\dot{\varphi} - \dot{\psi}\theta) - m z_G(\dot{\varphi} - \dot{\psi}\theta)^2 \\
& \qquad\qquad - m z_G(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi)^2 - Y_{\dot{r}}(\dot{\varphi} - \dot{\psi}\theta)(\dot{\psi}\cos\varphi \\
& \qquad\qquad - \dot{\theta}\sin\varphi) - m_L U(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi) \\
& K_\beta\beta + K_p(\dot{\varphi} - \dot{\psi}\theta) + K_r(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi) + K_{\delta_r}\delta_r - W z_G\sin\varphi \\
& \qquad = J_x(\ddot{\varphi} - \ddot{\psi}\theta - \dot{\theta}\dot{\psi}) - I_{xz}(\ddot{\psi}\cos\varphi - \dot{\psi}\dot{\theta}\theta\cos\varphi - \dot{\psi}\dot{\varphi}\sin\varphi
\end{aligned}
$$

(54) Contd.

$$- \ddot{\theta} \sin \varphi - \dot{\theta}\dot{\varphi} \cos \varphi) + mz_G V\dot{\beta}$$

$$+ (J_z - J_y)(\dot{\psi} \cos \varphi - \dot{\theta} \sin \varphi)(\dot{\psi} \sin \varphi + \dot{\theta} \cos \varphi)$$

$$- I_{xz}(\dot{\varphi} - \dot{\psi}\theta)(\dot{\psi} \sin \varphi + \theta \cos \varphi) + V(Y_{\dot{r}} + Z_{\dot{q}})\beta$$

$$(\dot{\psi} \sin \varphi + \dot{\theta} \cos \varphi) + mz_G V\alpha(\dot{\varphi} - \dot{\psi}\theta)$$

$$+ V(Y_{\dot{r}} + Z_{\dot{q}})\alpha(\dot{\psi} \cos \varphi - \dot{\theta} \sin \varphi)$$

$$- mz_G V(\dot{\psi} \cos \varphi - \dot{\theta} \sin \varphi)$$

$$M_\alpha \alpha + M_q(\dot{\psi} \sin \varphi + \dot{\theta} \cos \varphi) + M_{\dot{\alpha}}\dot{\alpha} + M_{\delta_e}\delta_e + Bx_B \cos \varphi - Wz_G\theta$$

$$= J_y(\ddot{\psi} \sin \varphi - \dot{\psi}\dot{\theta}\theta \sin \varphi + \dot{\psi}\dot{\varphi} \cos \varphi + \ddot{\theta} \cos \varphi - \dot{\theta}\dot{\varphi} \sin \varphi)$$

$$+ mz_G\dot{u} + (J_x - J_z)(\dot{\varphi} - \dot{\psi}\theta)(\dot{\psi} \cos \varphi - \dot{\theta} \sin \varphi)$$

$$+ I_{xz}(\dot{\varphi} - \dot{\psi}\theta)^2 - I_{xz}(\dot{\psi} \cos \varphi - \dot{\theta} \sin \varphi)^2 + N_{\dot{\beta}}\beta(\dot{\varphi} - \dot{\psi}\theta)$$

$$+ mz_G V\alpha(\dot{\psi} \sin \varphi + \dot{\theta} \cos \varphi) + mz_G V\beta(\dot{\psi} \cos \varphi - \dot{\theta} \sin \varphi)$$

$$N_\beta \beta + N_p(\dot{\varphi} - \dot{\psi}\theta) + N_r(\dot{\psi} \cos \varphi - \dot{\theta} \sin \varphi) + N_{\dot{\beta}}\dot{\beta}$$

$$+ N_{\delta_r}\delta_r - Bx_B \sin \varphi$$

$$= J_z(\ddot{\psi} \cos \varphi - \dot{\psi}\dot{\theta}\theta \cos \varphi - \dot{\psi}\dot{\varphi} \sin \varphi - \ddot{\theta} \sin \varphi$$

$$- \dot{\theta}\dot{\varphi} \cos \varphi) - I_{xz}(\ddot{\varphi} \quad \ddot{\psi}\theta - \dot{\theta}\dot{\psi})$$

$$+ (J_y - J_x)(\dot{\varphi} - \dot{\psi}\theta)(\dot{\psi} \sin \varphi + \dot{\theta} \cos \varphi)$$

$$- M_{\dot{\alpha}}(\dot{\varphi} - \dot{\psi}\theta)\alpha + I_{xz}(\dot{\psi} \cos \varphi - \dot{\theta} \sin \varphi)(\dot{\psi} \sin \varphi$$

$$+ \dot{\theta} \cos \varphi)$$

These equations are greatly simplified if terms involving products of velocities may be neglected. Dropping such terms, Eq. 54 becomes

$$(55)\quad X_o + X_u u + T_x - (W - B)\theta = m_L\dot{u} + mz_G(\ddot{\psi}\sin\varphi + \ddot{\theta}\cos\varphi)$$

$$Y_\beta\beta + Y_p(\dot{\varphi} - \dot{\psi}\theta) + Y_r(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi)$$

$$+ Y_{\dot{r}}(\ddot{\psi}\cos\varphi - \ddot{\theta}\sin\varphi) + Y_{\delta_r}\delta_r + (W - B)\sin\varphi$$

$$= -m_T V\beta - mz_G(\ddot{\varphi} - \ddot{\psi}\theta) + m_L V(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi)$$

$$Z_\alpha\alpha + Z_q(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi) + Z_{\dot{q}}(\ddot{\psi}\sin\varphi + \ddot{\theta}\cos\varphi)$$

$$+ Z_{\delta_e}\delta_e + (W - B)\cos\varphi$$

$$= m_T V\dot{\alpha} - m_L V(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi)$$

$$K_\beta\beta + K_p(\dot{\varphi} - \dot{\psi}\theta) + K_r(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi) + K_{\delta_r}\delta_r - Wz_G\sin\varphi$$

$$= J_x(\ddot{\varphi} - \ddot{\psi}\theta) - I_{xz}(\ddot{\psi}\cos\varphi - \ddot{\theta}\sin\varphi) + mz_G V\dot{\beta}$$

$$+ mz_G V(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi)$$

$$M_\alpha\alpha + M_q(\dot{\psi}\sin\varphi + \dot{\theta}\cos\varphi) + M_{\dot{\alpha}}\dot{\alpha} + M_{\delta_e}\delta_e + Bx_B\cos\varphi - Wz_G\theta$$

$$= J_y(\ddot{\psi}\sin\varphi + \ddot{\theta}\cos\varphi) + mz_G\dot{u}$$

$$N_\beta\beta + N_p(\dot{\varphi} - \dot{\psi}\theta) + N_r(\dot{\psi}\cos\varphi - \dot{\theta}\sin\varphi)$$

$$+ N_{\dot{\beta}}\dot{\beta} + N_{\delta_r}\delta_r - Bx_B\sin\varphi$$

$$= J_z(\ddot{\psi}\cos\varphi - \ddot{\theta}\sin\varphi) - I_{xz}(\ddot{\varphi} - \ddot{\psi}\theta)$$

where it has been assumed that $U_o \approx V$.

If $\varphi = I_{xz} = z_G = Y_p = N_p = 0$, and the forward velocity is constant, Eq. 55 may be reduced to two sets of equations, the yaw equations and the pitch equations. The yaw equations are

$$(56)\quad Y_\beta\beta + Y_r\dot{\psi} + Y_{\dot{r}}\ddot{\psi} + Y_{\delta_r}\delta_r = -m_T V\dot{\beta} + m_L V\dot{\psi}$$

$$N_\beta\beta + N_r\dot{\psi} + N_{\dot{\beta}}\dot{\beta} + N_{\delta_r}\delta_r = J_z\ddot{\psi}$$

The pitch equations are

$$(57)\qquad Z_\alpha \alpha + Z_q \dot{\theta} + Z_{\dot{q}} \ddot{\theta} + Z_{\delta_e} \delta_e + (W - B) = m_T V \dot{\alpha} - m_L V \dot{\theta}$$

$$M_\alpha \alpha + M_q \dot{\theta} + M_{\dot{\alpha}} \dot{\alpha} + M_{\delta_e} \delta_e + B x_B = J_y \ddot{\theta}$$

It is sometimes convenient to change the origin of the body-fixed coordinate system. Suppose the center of gravity is to be shifted, for example. It is then desirable to write the equations with the new center of gravity as the origin. Hydrodynamic forces, which depend only on the exterior shape of the body, are unchanged. There will be a change in the hydrodynamic coefficients, however, because of the change in angle of attack at the new origin. Suppose the origin to be shifted forward on the longitudinal axis a distance Δ, and consider the hydrodynamic force and moment to be

$$(58)\qquad Y = Y_{2v} v + Y_{2r} r + Y_{2\delta_r} \delta_r$$

$$N = N_{2v} v + N_{2r} r + N_{2\delta_r} \delta_r$$

the lateral velocity at the new origin is

$$(59)\qquad v^* = v + \Delta r$$

The moment around the new origin is

$$(60)\qquad N^* = N - Y\Delta$$

In terms of v^*, then, the force and moment at the new origin are

$$(61)\qquad Y = Y_{2v} v^* + (Y_{2r} - \Delta Y_{2v}) r + Y_{\delta_r} \delta_r$$

$$N^* = (N_{2v} - \Delta Y_{2v}) v^* + \left[N_{2r} - \Delta Y_{2r} - \Delta (N_{2v} - \Delta Y_{2v}) \right] r$$

$$+ (N_{2\delta_r} - \Delta Y_{2\delta_r}) \delta_r$$

Hence

$$(62)\qquad Y_{2v}^* = Y_{2v}$$

$$Y_{2r}^* = Y_{2r} - \Delta Y_{2v}$$

$$Y_{2\delta r}^* = Y_{2\delta r}$$

(62) Contd.

$$N_{2v}^* = N_{2v} - \Delta Y_{2v}$$

$$N_{2r}^* = N_{2r} - \Delta Y_{2r} - \Delta(N_{2v} - \Delta Y_{2v})$$

$$N_{2\delta r}^* = N_{2\delta r} - \Delta Y_{2\delta r}$$

Substitution of Eq. 59 into the expression for T_f (Eq. 36) gives

(63)

$$a_{66}^* = a_{66} - 2\Delta a_{26} + a_{22}\Delta^2$$

$$a_{26}^* = a_{26} - a_{22}\Delta$$

The yaw equations (Eq. 56) become

(64)

$$Y_\beta^*\beta^* + Y_r^*\dot{\psi} + Y_{\dot{r}}^*\ddot{\psi} + Y_{\delta_r}^*\delta_r = -m_T V\dot{\beta}^* + m_L V\dot{\psi}$$

$$N_\beta^*\beta^* + N_r^*\dot{\psi} + N_{\dot{\beta}}^*\dot{\beta}^* + N_{\delta_r}^*\delta_r = J_z^*\ddot{\psi}$$

where

(65)

$$Y_\beta^* = Y_\beta$$

$$Y_r^* = Y_r + \frac{\Delta}{V}Y_\beta$$

$$Y_{\delta_r}^* = Y_{\delta_r}$$

$$Y_{\dot{r}}^* = -a_{26}^*$$

$$N_\beta^* = N_\beta - \Delta Y_\beta$$

$$N_r^* = N_r - \Delta Y_r + \frac{\Delta}{V}(N_\beta - \Delta Y_\beta) + a_{11}\Delta V$$

$$N_{\dot{\beta}}^* = a_{26}^* V$$

$$N_{\delta_r}^* = N_{\delta_r} - \Delta Y_{\delta_r}$$

$$J_z^* = I_z^* + a_{66}^*$$

Similar transformations may be carried out for the pitch equations (Eq. 57). The new pitch equations are

$$(66) \quad Z_\alpha{}^*\alpha^* + Z_q{}^*\dot{\theta} + Z_{\dot{q}}{}^*\ddot{\theta} + Z_{\delta_e}{}^*\delta_e + (W - B) = m_T V\dot{\alpha}^* - m_L V\dot{\theta}$$

$$M_\alpha{}^*\alpha^* + M_q\dot{\theta} + M_{\dot{\alpha}}{}^*\dot{\alpha}^* + M_{\delta_e}{}^*\delta_e + Bx_B = J_y{}^*\ddot{\theta}$$

where

$$(67) \quad Z_\alpha{}^* = Z_\alpha$$

$$Z_q{}^* = Z_q + \frac{\Delta}{V}Z_\alpha$$

$$Z_{\delta_e}{}^* = Z_{\delta_e}$$

$$Z_{\dot{q}}{}^* = -a_{35}{}^*$$

$$M_\alpha{}^* = M_\alpha + \Delta Z_\alpha$$

$$M_q{}^* = M_q + \Delta Z_q + \frac{\Delta}{V}(M_\alpha + \Delta Z_\alpha) + a_{11}V\Delta$$

$$M_{\dot{\alpha}}{}^* = -a_{35}{}^*V$$

$$M_{\delta_e}{}^* = M_{\delta_e} + \Delta Z_{\delta_e}$$

$$J_y{}^* = I_y{}^* + a_{55}{}^*$$

$$a_{55}{}^* = a_{55} + 2\Delta a_{35} + a_{33}\Delta^2$$

$$a_{35}{}^* = a_{35} + a_{33}\Delta$$

SOURCES OF HYDRODYNAMIC COEFFICIENTS

MODEL TESTS AS SOURCES OF HYDRODYNAMIC COEFFICIENTS

Model tests may be classified into three types: static tests, rotating-arm tests, and forced-oscillation tests.

Static Tests

In static tests the model may be towed through a tank or placed in a water tunnel at a fixed angle of attack. The resultant forces and moments are then measured. Suppose the model to be towed through a tank at a fixed angle of sideslip β, and let the measured force and moment be, respectively, Y_M and N_M. From Eq.56

$$Y_M + Y_\beta \beta + Y_{\delta_r} \delta_r = 0 \tag{68}$$

$$N_M + N_\beta \beta + N_{\delta_r} \delta_r = 0$$

By repeating the experiment at different sideslip angles, rudder settings, and velocities Y_β, Y_{δ_r}, N_β, and N_{δ_r} may be determined as functions of β, δ_r, and velocity. By rolling the model through 90 degrees, the coefficients Z_α, Z_{δ_e}, M_α, and M_{δ_e} may be determined in a similar manner.

The drag X_o may be determined by measuring the force component along the longitudinal axis. It is found that for the range of velocities in which most torpedoes operate X_o may be expressed in terms of a dimensionless coefficient X_o' and the velocity as

$$X_o = 1/2 \rho A X_o' V^2 \tag{69}$$

where A is a characteristic area.

The coefficient X_u may be obtained from Eq. 2 by differentiating it with respect to V. Thus

$$X_u = \rho A X_o' V \tag{70}$$

Rotating-Arm Tests

In rotating-arm tests the model is towed in a circular path at fixed angular velocities and at a fixed angle of attack. For this condition the motion equations (Eq. 56) become

$$Y_M + Y_\beta \beta + (Y_r - m_L V)\dot{\psi} + Y_{\delta_r} \delta_r = 0 \tag{71}$$

$$N_M + N_\beta \beta + N_r \dot{\psi} + N_{\delta_r} \delta_r = 0$$

From these equations the coefficients Y_r and N_r may be determined. The coefficients $Y_{\dot{r}}$ and $M_{\dot{\alpha}}$ cannot be measured with this type of test since an accelerated motion is required. Measurement of these coefficients may be accomplished by forced oscillation tests.

The rotating-arm test has the advantage, however, of making possible the determination of the nonlinearity of the hydrodynamic coefficients. For example, M_α may be determined as a nonlinear function of α by measurement of the moment M over a range of angle of attack α. For a more complete discussion of the nonlinearities of the coefficients see Ref. 4.

Forced-Oscillation Tests

In one such test the model is supported by means of a shaft in a water tunnel. The shaft is made to oscillate through the application of a sinusoidal torque applied through a spring. Measurements of amplitude and phase then permit the computation of the hydrodynamic coefficients.

The model is placed in the water tunnel as shown in Fig. 1.

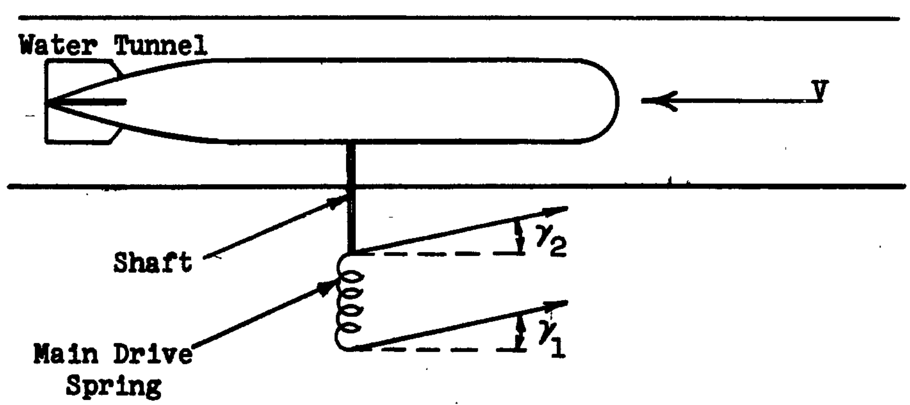


FIG. 1

Let

K_1 = spring constant of main drive spring

K_2 = spring constant of shaft

γ_1 = input displacement angle

γ_2 = output displacement angle

Define γ_1 and γ_2 positive in the same sense as the yaw angle ψ. Two or three support positions may be used in the test. The hydrodynamic coefficients will be defined about one of these positions. Suppose three positions to be used, and let the hydrodynamic coefficients be defined about the center support position. The motion equations of the model are

(72) $$I_z\ddot{\psi} = N_\beta\beta + N_r\dot{\psi} + N_{\dot\beta}\dot\beta + N_{\dot r}\ddot{\psi} + N_o$$

(73) $$mV(\dot{\psi} - \dot\beta) = Y_\beta\beta + Y_r\dot{\psi} + Y_{\dot\beta}\dot\beta + Y_{\dot r}\ddot{\psi} + Y_o$$

where N_o is the moment about the central support position applied to the model through the shaft, and Y_o is the side force applied to the model by the shaft. When the rotation is about the center axis,

(74) $$N_o = -K_2(\psi - \gamma_2) = -K_1(\gamma_2 - \gamma_1)$$

When the rotation is about the forward axis, the applied moment is given by

(75) $$N_o = -K_2(\psi - \gamma_2) + Y_o\Delta$$

where Δ is the distance between the center and forward support positions.

The angle of sideslip at the center support position is equal to ψ when the rotation is about the center support. When the model is rotated about the forward support, the angle of sideslip is given by

(76) $$\beta = \psi + \frac{\Delta}{V}\dot{\psi}$$

Hence, when the forward support position is the center of rotation,

(77) $$Y_o = -m\Delta\ddot{\psi} - Y_\beta\left(\psi + \frac{\Delta}{V}\dot{\psi}\right) - Y_r\dot{\psi} - Y_{\dot\beta}\left(\dot{\psi} + \frac{\Delta}{V}\ddot{\psi}\right) - Y_{\dot r}\ddot{\psi}$$

and the applied moment is

(78) $$N_o = -K_2(\psi - \gamma_2) - \Delta\left[\left(m\Delta + Y_{\dot r} + \frac{\Delta}{V}Y_{\dot\beta}\right)\ddot{\psi} + \left(Y_r + \frac{\Delta}{V}Y_\beta + Y_{\dot\beta}\right)\dot{\psi} + Y_\beta\psi\right]$$

Equation 72 then becomes

(79) $$\left[I_z - N_{\dot r} - \frac{\Delta}{V}N_{\dot\beta} + \Delta\left(m\Delta + Y_{\dot r} + \frac{\Delta}{V}Y_{\dot\beta}\right)\right]\ddot{\psi} - \left[N_r + N_{\dot\beta} + \frac{\Delta}{V}N_\beta - \Delta\left(Y_r + Y_{\dot\beta} + \frac{\Delta}{V}Y_\beta\right)\right]\dot{\psi} - \left[N_\beta - \Delta Y_\beta\right]\psi = -K_2(\psi - \gamma_2)$$

Let

$$(80) \qquad P = I_z - N_{\dot{r}} - \frac{\Delta}{V} N_{\dot{\beta}} + \Delta(m\Delta + Y_{\dot{r}} + \frac{\Delta}{V} Y_{\dot{\beta}})$$

$$-Q = N_r + N_{\dot{\beta}} + \frac{\Delta}{V} N_\beta - \Delta(Y_r + Y_{\dot{\beta}} + \frac{\Delta}{V} Y_\beta)$$

$$R = N_\beta - Y_\beta \Delta$$

Then Eq. 79 becomes

$$(81) \qquad P\ddot{\psi} + Q\dot{\psi} - R\psi = -K_2(\psi - \gamma_2)$$

When the center support is the center of rotation $\Delta = 0$. The coefficients for this case will be denoted by P_c, Q_c, and R_c. The coefficients for the forward support position will be denoted by P_F, Q_F, and R_F.

The equation relating ψ, γ_1, and γ_2 as obtained from Eq. 74 is

$$(82) \qquad \psi = (1 + \frac{K_1}{K_2})\gamma_2 - \frac{K_1}{K_2}\gamma_1$$

Let the input displacement angle γ_1 be

$$(83) \qquad \gamma_1 = A_1 \sin \omega t$$

The measured output is

$$(84) \qquad \gamma_2 = A_2 \sin (\omega t - \epsilon)$$

$$= A_2 \cos \epsilon \sin \omega t - A_2 \sin \epsilon \cos \omega t$$

Hence, from Eq. 83,

$$(85) \qquad \psi = \left[(1 + \frac{K_1}{K_2})A_2 \cos \epsilon - \frac{K_1}{K_2} A_1 \right] \sin \omega t$$

$$- (1 + \frac{K_1}{K_2})A_2 \sin \epsilon \cos \omega t$$

Let

(86) $$\left(1 + \frac{K_1}{K_2}\right)A_2 \cos \epsilon - \frac{K_1}{K_2} A_1 = a, - \left(1 + \frac{K_1}{K_2}\right)A_2 \sin \epsilon = b$$

Solution of Eq. 81 then yields

(87) $$Q = \frac{K_1 A_1 A_2 \sin \epsilon}{\omega(a^2 + b^2)}$$

(88) $$R + \omega^2 P = K_2 - K_2 A_2 \left[\frac{\left(1 + \frac{K_1}{K_2}\right) A_2 - \frac{K_1}{K_2} A_1 \cos \epsilon}{a^2 + b^2}\right]$$

Since the right-hand sides of Eq. 87 and 88 contain only measurable quantities, it is possible to determine the combinations of hydrodynamic coefficients of Eq. 80. Thus

(89) $$P_c = I_{zc} - N_{\dot{r}}$$

$$-Q_c = N_r + N_{\dot{\beta}}$$

$$R_c = N_\beta$$

where I_{zc} is the moment of inertia about the center support when the model is rotated about the center support position. Rotation about the forward support position yields

(90) $$P_F = I_{zF} - N_{\dot{r}} - \frac{\Delta}{V} N_\beta + \Delta\left(m_F\Delta + Y_{\dot{r}} + \frac{\Delta}{V} Y_\beta\right)$$

$$-Q_F = N_r + N_{\dot{\beta}} + \frac{\Delta}{V} N_\beta - \Delta\left(Y_r + Y_{\dot{\beta}} + \frac{\Delta}{V} Y_\beta\right)$$

$$R_F = N_\beta - \Delta Y_\beta$$

where I_{zF} is the moment of inertia about the center support position when the center of rotation is the forward support. It may differ from I_{zc} because the mass of the model may be changed slightly in changing the center of rotation. The mass of the model when the forward support position is used is denoted by m_F. With three support positions Eq. 89 and 90 yield all the hydrodynamic coefficients if it is assumed that

$$Y_{\dot{r}} = -\frac{1}{V} N_{\dot{\beta}}$$

If only two support positions are used it is necessary to estimate one of them, and $Y_{\dot{\beta}}$ is the most convenient to estimate.

Another type of forced oscillation test may be used to determine the lift coefficient. The model is made to oscillate laterally in the water tunnel by means of a sinusoidal force applied through a drive spring as shown in Fig. 2.

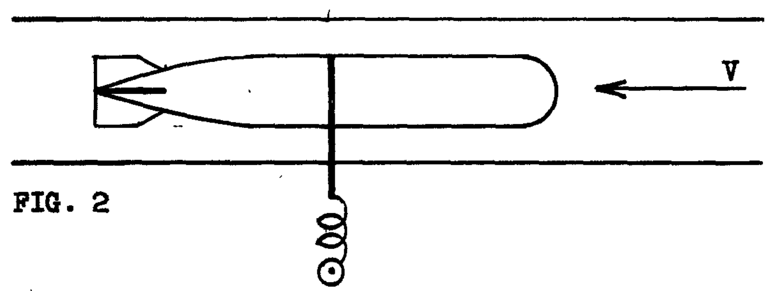


FIG. 2

Let displacement of the lower end of the spring be ξ and the displacement of the upper end η. The force on the model is

(91) $$Y_o = K(\xi - \eta)$$

where K is the constant of the spring. The motion equation of the model is

(92) $$(mV + Y_{\dot{\beta}})\dot{\beta} + Y_\beta \beta + Y_o = 0$$

The angle of sideslip is

(93) $$\beta = -\frac{\dot{\eta}}{V}$$

Hence, using Eq. 93, Eq. 92 becomes

(94) $$(m + \frac{Y_{\dot{\beta}}}{V})\ddot{\eta} + (\frac{Y_\beta}{V})\dot{\eta} + K\eta = K\xi$$

If the measured response to an input,

(95) $$\xi = A_1 \sin \omega t$$

is

(96) $$\epsilon = A_2 \sin (\omega t - \epsilon)$$

Eq. 94 may be solved for Y_β and $Y_{\dot\beta}$ as

(97)
$$Y_\beta = \frac{KV}{\omega}\frac{A_1}{A_2}\sin\epsilon$$

$$Y_{\dot\beta} = \frac{KV}{\omega^2}\left(1 - \frac{A_1}{A_2}\cos\epsilon\right) - mV$$

OTHER SOURCES OF HYDRODYNAMIC COEFFICIENTS

At the present time model tests carried out in water tunnels or towing tanks are the most dependable sources of hydrodynamic coefficients. For preliminary design purposes, however, the estimation of the hydrodynamic characteristics of a torpedo yet to be built is essential. This problem will become more acute when torpedoes of higher speed and more complicated trajectories are built. It will then be necessary to specify a body shape and a tail configuration that will permit desired performance while the torpedo is still in the drawing-board phase of development. Attempts are being made to estimate coefficients on the basis of empirical data and hydrodynamic theory. The aim of this work is to determine the hydrodynamic coefficients given a body shape and tail configuration, or, specifying coefficients, to construct a body and tail configuration having the desired hydrodynamic coefficients. Model tests would then be only a check on the preliminary values if these attempts are successful.

The mass accession terms of the motion equations are usually estimated since they cannot be determined from static or rotating-arm tests on models. It is customary to evaluate these terms by assuming the torpedo to be an ellipsoid of revolution. The coefficients a_{ij} of Eq. 36 are then given by

$$a_{11} = k_1 m_f$$

$$a_{22} = a_{33} = k_2 m_f$$

$$a_{44} = 0$$

$$a_{55} = a_{66} = k' I_f$$

$$a_{26} = a_{35} = 0$$

where m_f is the mass of the displaced fluid and I_f is the moment of inertia of the displaced fluid about a minor axis of the ellipsoid. The k_1, k_2, and k' are Lamb's coefficients (see Ref.8). It has been assumed that the origin is at the center of the ellipsoid. If the origin is forward of the center a distance Δ the coefficients a_{ij} are given by

$$a_{11} = k_1 m_f$$

$$a_{22} = a_{33} = k_2 m_f$$

$$a_{44} = 0$$

$$a_{55} = a_{66} = k' I_f + k_2 m_f \Delta^2$$

$$a_{35} = -a_{26} = k_2 m_f \Delta$$

A check on the estimated coefficients may be obtained from free-flight tests of instrumented torpedoes. When records of pitch, yaw, roll, depth, and control-surface deflections are obtained from full-scale torpedoes in free flight, a comparison of the recorded response with the response computed from the equations of motion may be made. This avenue of approach has not as yet been fully exploited, and much work remains to be done before techniques are developed for determining hydrodynamic coefficients from free-flight records.

FIELDS OF APPLICATION OF MOTION EQUATIONS

The nonlinear motion equations (Eq. 54) are very complex. Moreover, in order that they be exact, it is necessary that the hydrodynamic coefficients of the left-hand members of the equations be considered as nonlinear functions of the velocity components. It is improbable, therefore, that the complete equations in this form will ever be of great use to engineers. The partially linearized equations (Eq. 55), on the other hand, may be solved without great difficulty with the use of an analog computer. In cases where they are valid, these simplified equations can provide valuable information about the trajectory of a torpedo. In most studies that have been made up to the present time the pitch and yaw equations (Eq. 56 and 57) have been deemed sufficient.

SOLUTION OF STEADY-STATE EQUATIONS

The motion equations (Eq. 54) may be solved if a steady-state motion exists, such as trimmed straight flight or a steady-state turn. Solution of the equations in these cases is useful because, in the one case, the trim pitch angle and elevator setting must be known in order to set the control system for straight flight; and in the other case, the change of depth in a steady-state turn may be determined.

Trim Flight

If the torpedo is moving in straight trimmed flight

$$\dot{u} = \ddot{\psi} = \dot{\psi} = \varphi = \dot{\varphi} = \dot{\theta} = \ddot{\theta} = \beta = \dot{\beta} = \dot{\alpha} = \delta_r = 0$$

and

$$\theta = \alpha$$

The motion equations (Eq. 54) become

(98)
$$X_o + T_x - (W - B)\alpha + X_u u = 0$$

$$Z_\alpha \alpha + Z_{\delta_e}\delta_e + (W - B) = 0$$

$$M_\alpha \alpha + M_{\delta_e}\delta_e + Bx_B - Wz_G\alpha = 0$$

Solving for the trim values of u, α, and δ_e gives

$$\delta_{e_1} = \frac{-Z_\alpha Bx_B + (W - B)(M_\alpha - Wz_G)}{Z_\alpha M_{\delta_e} - Z_{\delta_e}(M_\alpha - Wz_G)}$$

$$\alpha_1 = \frac{-Z_{\delta_e}Bx_B + M_{\delta_e}(W - B)}{Z_{\delta_e}(M_\alpha - Wz_G) - M_{\delta_e}Z_\alpha}$$

$$u_1 = \frac{X_o + T_x}{-X_u} + \frac{(W - B)\alpha_1}{X_u}$$

Steady-State Trim

If the rudder is given a steady deflection when the torpedo is in straight trimmed flight, a rolling moment will be produced because of the centrifugal force acting on the center of gravity of the torpedo, which lies below the longitudinal axis. Because of the resulting roll the rudders will cause the torpedo to spiral downward. However, the change in depth will cause the control system to function, giving an up elevator which will tend to reduce the depth error. When a steady state is attained, the torpedo will circle at a constant depth error with constant angles of roll, pitch, sideslip and attack, and a constant elevator deflection different from the trim value. Under these conditions

$$\dot{\psi} = \text{constant}$$

and

$$\dot{u} = \dot{\theta} = \dot{\varphi} = \ddot{\theta} = \ddot{\psi} = \ddot{\varphi} = \dot{\alpha} = \dot{\beta} = 0$$

The motion equations (Eq. 54) become

(99) $$X_o + X_u u + T_x - (W - B)\theta = m_T V(\alpha \sin\varphi + \beta\cos\varphi)\dot{\psi} + \dot{\psi}^2(-Z_{\dot{q}} \sin^2\varphi + Y_{\dot{r}} \cos^2\varphi) - m z_G \dot{\psi}^2 \theta \cos\varphi$$

$$Y_\beta \beta - Y_p \dot{\psi}\theta + Y_r \dot{\psi}\cos\varphi + Y_{\delta_r}\delta_r + (W - B)\sin\varphi = m_L U\dot{\psi}\cos\varphi + m z_G \dot{\psi}^2 \cos\varphi \sin\varphi + m_T V\alpha\dot{\psi}\theta - Z_{\dot{q}}\dot{\psi}^2\theta \sin\varphi$$

$$Z_\alpha \alpha + Z_q \dot{\psi}\sin\varphi + Z_\delta \delta_e + (W - B)\cos = m_T V\dot{\psi}\theta\beta - m z_G \dot{\psi}^2\theta^2 - m z_G \dot{\psi}^2 \sin^2\varphi + Y_{\dot{r}}\dot{\psi}^2\theta\cos\varphi - m_L U\dot{\psi}\sin\varphi$$

(99) Contd.

$$K_\beta \beta - K_p \dot\psi \theta + K_r \dot\psi \cos\varphi + K_{\delta_r}\delta_r - W z_G \sin\varphi$$

$$= (J_z - J_y)\dot\psi^2 \cos\varphi \sin\varphi + I_{xz}\dot\psi^2\theta \sin\varphi$$

$$+ V\dot\psi(Y_{\dot r} + Z_{\dot q})(\beta \sin\varphi + \alpha \cos\varphi) - m z_G U \dot\psi \cos\varphi$$

$$- m z_G V \alpha \dot\psi \theta$$

$$M_\alpha \alpha + M_q \dot\psi \sin\varphi + M_{\delta_e}\delta_e + B x_B \cos\varphi - W z_G \theta$$

$$= (J_z - J_x)\dot\psi^2\theta \cos\varphi + I_{xz}\dot\psi^2\theta^2 - I_{xz}\dot\psi^2 \cos^2\varphi$$

$$- N_{\dot\beta}\beta\dot\psi\theta + m z_G V \dot\psi(\alpha \sin\varphi + \beta \cos\varphi)$$

$$N_\beta \beta - N_p \dot\psi \theta + N_r \dot\psi \cos\varphi + N_{\delta_r}\delta_r - B x_B \sin\varphi$$

$$= (J_x - J_y)\dot\psi^2\theta \sin\varphi + M_{\dot\alpha}\alpha\dot\psi\theta + I_{xz}\dot\psi^2 \cos\varphi \sin\varphi$$

$$\theta = \alpha \cos\varphi - \beta \sin\varphi$$

These equations are to be solved for $\dot\psi$, φ, β, α, θ, u, and δ_e. A numerical method of successive approximation is probably most convenient to use. A first approximation may be obtained by linearizing Eq. 99 in all variables except φ. A first approximation to φ is given by

(100) $$\varphi = \tan^{-1}\frac{V\dot\psi}{g}$$

The following equations then yield first approximation to $\dot\psi$ and β.

(101) $$Y_\beta \beta + Y_r \dot\psi \cos\varphi + Y_{\delta_r}\delta_r + (W - B)\sin\varphi = m_L V \dot\psi \cos\varphi$$

$$N_\beta \beta + N_r \dot\psi \cos\varphi + N_{\delta_r}\delta_r - B x_B \sin\varphi = 0$$

Using the first approximations to $\dot\psi$ and β obtained from Eq. 101 Eq. 102 may be used to yield first approximations to α, δ_e, and

$$(102)\quad Z_\alpha \alpha + Z_q \dot{\psi} \sin \varphi + Z_{\delta_e} \delta_e + (W - B) \cos \varphi = - m_L V \dot{\psi} \sin \varphi$$

$$M_\alpha \alpha + M_q \dot{\psi} \sin \varphi + M_{\delta_e} \delta_e + B x_B \cos \varphi - W z_G \theta = 0$$

$$\theta = \alpha \cos \varphi - \beta \sin \varphi$$

If more accurate solutions are desired a numerical method of successive approximations may be used starting with the first approximations given by Eq. 100, 101, and 102 (see, for example, Numerical Calculus, W. E. Milne, Princeton University Press, 1949).

Depth error in the turn may be calculated by considering the depth error necessary to yield the elevator setting given by Eq. 99.

STABILITY OF CONTROLLED TORPEDOES

The behavior of a torpedo in the water is a function not only of its hydrodynamic characteristics but also of its internal control system. The complete system must be considered before it can be decided whether a torpedo is capable of the performance that is required. Study of the complete system, comprised of hydrodynamic characteristics and control system, is usually termed "stability analysis". It is not the purpose of this report to discuss all the methods by which stability analyses may be undertaken, but it seems appropriate to describe the manner in which the motion equations enter into the problem. A control system for a torpedo contains devices that can detect the position or attitude of the torpedo or their rates of change. Signals from these devices are used to control the action of elevators or rudders which produce changes in the trajectory of the torpedo. Thus the controlled torpedo constitutes a feed-back system, or servomechanism, and standard analysis techniques from the theory of servomechanisms are applicable (see Ref. 9).

A simple example will be used to show how a control system may be analyzed. Assume that the trajectory of a torpedo is in a horizontal plane and that the torpedo does not roll. The motion equations are given (Eq. 56) as

$$(103)\quad - m_T V \dot{\beta} = Y_\beta \beta + (Y_r - m_L V) \dot{\psi} + Y_{\dot{r}} \ddot{\psi} + Y_{\delta_r} \delta_r$$

$$(104)\quad J_z \ddot{\psi} = N_\beta \beta + N_r \dot{\psi} + N_{\dot{\beta}} \dot{\beta} + N_{\delta_r} \delta_r$$

The rudder is controlled by a signal obtained from a device sensitive to direction, such as a gyroscope. It will be assumed that the rudder deflection is proportional to the difference between the torpedo direction ψ and a reference direction ψ_s. A time lag T_ℓ is introduced in the equation to represent the delay in the actuator that operates the rudder.

Thus

$$(105) \qquad \left(1 + T_\ell \frac{d}{dt}\right)\delta_r = K_\psi(\psi - \psi_s)$$

Equations 103, 104, and 105 may be solved for ψ in terms of ψ_s. Usually ψ_s is a constant direction. Depending on K_ψ and the hydrodynamics, the motion of the torpedo will be either stable or unstable. It is said to be stable if transient oscillations are eventually damped out and the torpedo assumes the direction ψ_s. The motion is unstable if the oscillations continually increase in amplitude. It is the aim of the designer to choose a K_ψ that will result in a fast well-damped response to a disturbance.

Even a simple system such as that of the example cited above requires long tedious computation if it is to be solved without recourse to mechanical or electrical computers. A great saving in time and effort is achieved by the use of such aids to computation. This type of problem is particularly amenable to solution with the use of an analog computer such as the REAC (Ref. 10). Appendix G shows how Eq. 103, 104, and 105 are solved on the REAC at the U. S. Naval Ordnance Test Station. The REAC is capable of solving much more complicated problems, but the principle of operation is demonstrated rather well by this example.

Appendix A

NOMENCLATURE

a_{ij}	Components of apparent mass tensor
$\underline{B}$	Buoyancy of torpedo
$\underline{F}$	Force acting on torpedo
$\underline{F_1}$	Force on torpedo predicted from potential flow
$\underline{F_2}$	$\underline{F} - \underline{F_1}$
$\underline{f}$	Force per unit volume acting on fluid
g	Acceleration of gravity
$\underline{G_b}$	Linear momentum of torpedo
$\underline{G_f}$	Linear momentum of fluid
$\underline{G}$	$\underline{G_f} + \underline{G_b}$
$\underline{H_b}$	Angular momentum of torpedo
$\underline{H_f}$	Angular momentum of fluid
$\underline{H}$	$\underline{H_f} + \underline{H_b}$
H_x, H_y, H_z	Components of $\underline{H}$ in body coordinates
I_x, I_y, I_z	Moments of inertia about x, y, z axes, respectively
I_{yz}, I_{xz}, I_{xy}	Products of inertia
J_x, J_y, J_z	Apparent moments of inertia about x, y, z axes, respectively
k	Viscosity coefficient of fluid
K_2, M_2, N_2	Components of $\underline{L_2}$ in body coordinates

$\underline{L}$	Moment acting on torpedo
$\underline{L}_1$	Moment on torpedo predicted from potential flow
$\underline{L}_2$	$\underline{L} - \underline{L}_1$
m	Mass of torpedo
m_L	Apparent longitudinal mass of torpedo
m_T	Apparent transverse mass of torpedo
$\underline{n}$	Unit normal to surface
p, q, r	Components of $\underline{\omega}$ in body coordinates
P	Pressure acting on fluid
$\underline{q}_1$	Velocity field
$\underline{T}$	Thrust of torpedo
T_x	Magnitude of thrust of torpedo
T_b	Kinetic energy of torpedo body
T_f	Kinetic energy of fluid
T	$T_b + T_f$
U, v, w	Components of $\underline{V}$ in body coordinates
$\underline{V}$	Velocity of torpedo
$\underline{W}$	Weight of torpedo
x, y, z	Body-fixed coordinates
x_o, y_o, z_o	Space-fixed coordinates
x_B, y_B, z_B	Components of buoyancy moment arm
x_G, y_G, z_G	Body coordinates of torpedo c. g.
X_2, Y_2, Z_2	Components of $\underline{F}_2$ in body coordinates

α	Angle of attack of torpedo
β	Angle of sideslip of torpedo
δ_e	Elevator deflection
δ_r	Rudder deflection
λ, μ, ν	Direction cosines of $\underline{n}$
ρ	Density of fluid
ϕ_1	Velocity potential function
ψ, θ, φ	Inertial reference angles of torpedo
$\underline{\omega}$	Angular velocity of torpedo

Appendix B

FORCE AND MOMENT ON TORPEDO PREDICTED FROM IDEAL FLUID

Consider the torpedo to be immersed in an ideal fluid of infinite extent, the fluid being at rest at infinity. Let the velocity field be denoted by $\underline{q}$. The total linear momentum of the fluid is given by

$$\underline{G}_f = \int_{\tau} \rho \underline{q} \, d\tau \tag{106}$$

where the integral is over the entire volume of the fluid. The momentum of the fluid must have a finite value since otherwise it would be implied that an infinite momentum had been imparted to the fluid by finite forces exerted for a finite time by the torpedo, and this is impossible. Now consider an element of the fluid occupying the volume $d\tau$. By Newton's second law the force $d\underline{F}$ acting on the element is

$$d\underline{F} = \frac{d}{dt}(\rho \underline{q} \, d\tau) \tag{107}$$

The total force acting on the fluid is obtained by integrating over the whole fluid. Then, if $\underline{F}_f$ denote the total force acting on the fluid,

$$\begin{aligned} \underline{F}_f &= \int_{\tau} \frac{d}{dt}(\rho \underline{q} \, d\tau) \\ &= \frac{d}{dt} \int_{\tau} \rho \underline{q} \, d\tau \\ &= \frac{d\underline{G}_f}{dt} \end{aligned} \tag{108}$$

The total force on the fluid is that exerted by the torpedo. Hence, by Newton's third law the force $\underline{F}_1$ on the torpedo is

$$\underline{F}_1 = -\underline{F}_f = -\frac{d\underline{G}_f}{dt} \tag{109}$$

In a similar manner the net moment acting on the fluid about a point in inertial space may be shown to be equal to the inertial time rate of change of the total angular momentum of the fluid. Then, also,

$$\underline{L}_1 = -\frac{d\underline{H}_f}{dt} \tag{110}$$

where

$$\underline{H}_f = \int_\tau \rho\, \underline{S} \times \underline{q} \;\; d\tau \tag{111}$$

$\underline{S}$ being the radius vector of the fluid element $\rho d\tau$.

Appendix C

FORCE AND MOMENT REFERRED TO A MOVING COORDINATE SYSTEM

The laws of dynamics state that the external force acting on a system is equal to the time rate of change of linear momentum of the system, and that the moment of this force about a point fixed in space is equal to the time rate of change of the angular momentum about this point. Let Oxyz be the body-fixed coordinate system of the torpedo, and let the velocity $\underline{V}$ of O have the components U, v, w on these axes. The angular velocity $\underline{\omega}$ about O has the components p, q, r. Let $\underline{G}$ be the linear momentum of the system and let $\underline{H}'$ be the angular momentum about a point O' fixed in inertial space. Let $\underline{H}$ be the angular momentum about O. The time rate of change of a quantity seen from O' will be denoted by the operator d/dt. The time rate of change as seen from the moving coordinate system Oxyz will be denoted by a dot placed over a symbol.

The force acting on the system is given by (see Ref. 6)

$$\underline{F} = \frac{d\underline{G}}{dt} = \dot{\underline{G}} + \underline{\omega} \times \underline{G} \tag{112}$$

The angular momentum $\underline{H}'$ about O' and the angular momentum $\underline{H}$ about O are related by

$$\underline{H}' = \underline{H} + \underline{S} \times \underline{G} \tag{113}$$

where $\underline{S}$ is the radius-vector from O' to O. Thus the moment $\underline{L}'$ about O' is given by

$$\underline{L}' = \frac{d\underline{H}'}{dt} = \frac{d\underline{H}}{dt} + \underline{S} \times \underline{F} + \underline{V} \times \underline{G} \tag{114}$$

since $\underline{V} = d\underline{S}/dt$. The moment $\underline{L}$ about O is given by

$$\underline{L} = \underline{L}' - \underline{S} \times \underline{F} \tag{115}$$

Hence,

$$\underline{L} = \dot{\underline{H}} + \underline{\omega} \times \underline{H} + \underline{V} \times \underline{G} \tag{116}$$

Appendix D

DERIVATION OF MOMENTUM FROM KINETIC ENERGY

The kinetic energy of an ideal fluid is given in Eq. 31 as

$$T = \frac{1}{2}\rho \int_S \varphi \frac{\partial \varphi}{\partial n} dS \tag{117}$$

where the integral is taken over the surface of the torpedo. The potential function has the form

$$\varphi = U\varphi_1 + v\varphi_2 + w\varphi_3 + p\varphi_4 + q\varphi_5 + r\varphi_6 \tag{118}$$

The velocity field $\underline{q}$ of the fluid is obtained from the velocity potential as

$$\underline{q} = -\nabla\varphi \tag{119}$$

The total linear momentum of the fluid is obtained by integrating the momenta of the mass elements. Thus

$$\underline{G}_f = -\rho\int_\tau \nabla\varphi \, d\tau \tag{120}$$

By means of the divergence theorem the volume integration may be expressed as an integration over the torpedo surface as

$$\underline{G}_f = \rho\int_S \varphi \, \underline{n} \, dS \tag{121}$$

where $\underline{n}$ is defined as in the section "Kinetic Energy of Ideal Fluid." The component of $\underline{G}$ in the x direction (using Eq. 24) is

$$\begin{aligned} G_{f_x} &= \rho\int_S \varphi\lambda \, dS \\ &= -\rho\int_S \varphi \frac{\partial\varphi_1}{\partial n} dS \end{aligned} \tag{122}$$

in view of Eq. 25. Using Eq. 118, Eq. 122 becomes

$$G_{f_x} = a_{11}U + a_{12}v + a_{13}w + a_{14}p + a_{15}q + a_{16}r \tag{123}$$

It is seen, then, that

$$G_{f_x} = \frac{\partial T_f}{\partial U} \tag{124}$$

It may be shown in a similar manner that the other components of $\underline{G_f}$ are

$$G_{f_y} = \frac{\partial T_f}{\partial v} \tag{125}$$

$$G_{f_z} = \frac{\partial T_f}{\partial w}$$

The total angular momentum of the fluid is given by

$$\underline{H_f} = -\rho \int_S \underline{S} \times \nabla\varphi \; d\tau \tag{126}$$

where $\underline{S}$ is the radius vector of the fluid element $\rho\, d\tau$. Now

$$\nabla\varphi \times \underline{S} = \nabla \times \varphi S - \varphi \nabla \times \underline{S} \tag{127}$$

$$= \nabla \times \varphi \underline{S}$$

since $\nabla \times \underline{S} = 0$

By the divergence theorem, then, the volume integral of Eq. 126 may be transformed to an integral over the surface of the torpedo as

$$\underline{H_f} = -\rho \int_S \varphi \, \underline{n} \times \underline{S} \; dS \tag{128}$$

The component of $\underline{H_f}$ in the x direction is therefore

$$H_{f_x} = -\rho \int_S \varphi(\mu z - \nu y) \; dS \tag{129}$$

$$= -\rho \int_S \varphi \frac{\partial \varphi}{\partial n} \; dS$$

with reference to Eq. 27. Using Eq. 118 and 32, Eq. 129 may be written

$$H_{f_x} = a_{14}U + a_{24}v + a_{34}w + a_{44}p + a_{45}q + a_{46}r \tag{130}$$

$$= \frac{\partial T_f}{\partial p}$$

Similarly,

$$H_{f_y} = \frac{\partial T_f}{\partial q} \tag{131}$$

$$H_{f_z} = \frac{\partial T_f}{\partial r}$$

The components of momentum of the torpedo body may be determined from the kinetic energy of the body. The linear momentum is defined by

$$G_b = \sum m_i(\underline{V} + \underline{\omega} \times \underline{r}_i) \tag{132}$$

and the angular momentum by

$$H_b = \sum m_i\underline{r}_i \times (\underline{V} + \underline{\omega} \times \underline{r}_i) \tag{133}$$

where the summation is over all particles m_i of the torpedo, and $\underline{r}_i$ is the radius vector from the origin to m_i.

Since

$$T_b = \frac{1}{2} \sum m_i(\underline{V} + \underline{\omega} \times \underline{r}_i)^2 \tag{134}$$

$$\frac{\partial T_b}{\partial U} = \sum m_i(\underline{V} + \underline{\omega} \times \underline{r}_i)\left(\frac{\partial \underline{V}}{\partial U} + \frac{\partial \underline{\omega}}{\partial U} \times \underline{r}_i + \underline{\omega} \times \frac{\partial \underline{r}_i}{\partial U}\right) \tag{135}$$

$$= \sum m_i(\underline{V} + \underline{\omega} \times \underline{r}_i)\cdot \underline{i}$$

$$= G_{b_x}$$

Similarly, $\frac{\partial T_b}{\partial v} = G_{b_y}$ and $\frac{\partial T_b}{\partial w} = G_{b_z}$

Differentiating T_b with respect to p yields

$$\frac{\partial T_b}{\partial p} = \sum m_i(\underline{V} + \underline{\omega} \times \underline{r}_i)\left(\frac{\partial \underline{V}}{\partial p} + \frac{\partial \underline{\omega}}{\partial p} \times \underline{r}_i + \underline{\omega} \times \frac{\partial \underline{r}_i}{\partial p}\right) \tag{136}$$

(136)
Contd.

$$= \sum m_i(\underline{V} + \underline{\omega} \times \underline{r_i})\cdot(\underline{i} \times \underline{r_i})$$

$$= \sum m_i r_i \times (\underline{V} + \underline{\omega} \times \underline{r_i})\cdot\underline{i}$$

$$= H_{bx}$$

Similarly,

$$\frac{\partial T_b}{\partial q} = H_{by}, \quad \frac{\partial T_b}{\partial r} = H_{bx}$$

Consequently, since $T = T_b + T_f$, $\underline{G} = \underline{G}_f + \underline{G}_b$, and $\underline{H} = \underline{H}_b + \underline{H}_f$,

(137)

$$\underline{G} = \frac{\partial T}{\partial U}\underline{i} + \frac{\partial T}{\partial v}\underline{j} + \frac{\partial T}{\partial w}\underline{k}$$

$$\underline{H} = \frac{\partial T}{\partial p}\underline{i} + \frac{\partial T}{\partial q}\underline{j} + \frac{\partial T}{\partial r}\underline{k}$$

Appendix E

RESOLUTION OF BUOYANCY AND GRAVITY FORCES AND MOMENTS ONTO BODY COORDINATES

The orientation of a torpedo with respect to a fixed inertial frame of reference is specified by an angle of yaw ψ, an angle of pitch θ, and an angle of roll φ. Let the inertial axes be x_o, y_o, z_o, and let the unit vectors in the direction of these axes be $\underline{i}_o$, $\underline{j}_o$, $\underline{k}_o$, respectively. The body axes at the start coincide with the inertial axes and are then rotated about the z_o axis through an angle ψ to coincide with axes x_1, y_1, z_1. Let the unit vectors in the directions of these axes be $\underline{i}_1$, $\underline{j}_1$, $\underline{k}_1$. Then

(138)
$$\underline{i}_1 = \underline{i}_o \cos\psi + \underline{j}_o \sin\psi$$
$$\underline{j}_1 = -\underline{i}_o \sin\psi + \underline{j}_o \cos\psi$$
$$\underline{k}_1 = \underline{k}_o$$

Now the body axes are rotated about the y_1 axis through an angle θ to coincide with axes x_2, y_2, z_2. Letting the unit vectors in the directions of these axes be $\underline{i}_2$, $\underline{j}_2$, $\underline{k}_2$,

(139)
$$\underline{i}_2 = \underline{i}_1 \cos\theta - \underline{k}_1 \sin\theta$$
$$\underline{j}_2 = \underline{j}_1$$
$$\underline{k}_2 = \underline{i}_1 \sin\theta + \underline{k}_1 \cos\theta$$

Finally, the body axes are rotated around the x_2 axis through an angle φ. Letting the body axes be x, y, z and the unit vectors in the directions of these axes be $\underline{i}$, $\underline{j}$, $\underline{k}$, respectively, yields

(140)
$$\underline{i} = \underline{i}_2$$
$$\underline{j} = \underline{j}_2 \cos\varphi + \underline{k}_2 \sin\varphi$$
$$\underline{k} = -\underline{j}_2 \sin\varphi + \underline{k}_2 \cos\varphi$$

Combining these transformations gives

$$\underline{i} = \underline{i}_o \cos\psi \cos\theta + \underline{j}_o \sin\psi \cos\theta - \underline{k}_o \sin\theta \tag{141}$$

$$\underline{j} = \underline{i}_o(\cos\psi \sin\theta \sin\varphi - \sin\psi \cos\varphi)$$
$$+ \underline{j}_o(\sin\psi \sin\theta \sin\varphi + \cos\psi \cos\varphi)$$
$$+ \underline{k}_o \cos\theta \sin\varphi$$

$$\underline{k} = \underline{i}_o(\sin\psi \sin\varphi + \cos\psi \sin\theta \cos\varphi)$$
$$+ \underline{j}_o(\sin\psi \sin\theta \cos\varphi - \cos\psi \sin\varphi)$$
$$+ \underline{k}_o \cos\theta \cos\varphi$$

or

$$\underline{i}_o = \underline{i} \cos\psi \cos\theta + \underline{j}(\cos\psi \sin\theta \sin\varphi - \sin\psi \cos\varphi) \tag{142}$$
$$+ \underline{k}(\sin\psi \sin\varphi + \cos\psi \sin\theta \cos\varphi)$$

$$\underline{j}_o = \underline{i} \sin\psi \cos\theta + \underline{j}(\cos\psi \cos\varphi + \sin\psi \sin\theta \sin\varphi)$$
$$+ \underline{k}(\sin\psi \sin\theta \cos\varphi - \cos\psi \sin\varphi)$$

$$\underline{k}_o = -\underline{i} \sin\theta + \underline{j} \cos\theta \sin\varphi + \underline{k} \cos\theta \cos\varphi$$

The weight and buoyancy forces act in the z_o direction. Hence the vector representing the gravity buoyancy force is

$$(W - B)\underline{k}_o = (W - B)(-\underline{i} \sin\theta + \underline{j} \sin\varphi \cos\theta + \underline{k} \cos\varphi \cos\theta) \tag{143}$$

The components of the gravity-buoyancy force may be read from this expression.

If $\underline{r}_B$ is the radius vector from the origin of the body coordinates to the center of buoyancy, the moment about the origin caused by the buoyancy force is

$$\underline{r}_B \times (-B\underline{k}_o) \tag{144}$$

Since $\underline{r} = \underline{i}x_B$ the moment due to buoyancy is

(145) $$\underline{j}Bx_B \cos\varphi \cos\theta - \underline{k}Bx_B \sin\varphi \cos\theta$$

Letting $\underline{r_G}$ be the radius vector from the origin to the center of gravity of the torpedo, the moment about the origin caused by gravity is

(146) $$\underline{r_G} \times (W\underline{k_o})$$

Since

$$\underline{r_G} = z_G\underline{k}$$

the gravity moment is

(147) $$-\underline{i}Wz_G \sin\varphi \cos\theta - \underline{j}Wz_G \sin\theta$$

Appendix F

RESOLUTION OF ANGULAR VELOCITY TO INERTIAL COORDINATES

The angular velocity of the torpedo has the components p, q, r in body coordinates. It is required that the angular velocity be expressed in inertial coordinates (ψ, θ, φ). It is noted that $\dot{\psi}$ is the angular velocity about the z_o axis, $\dot{\theta}$ is the angular velocity about the y_1 axis, and $\dot{\varphi}$ is the angular velocity about the x axis. Hence

$$(148) \qquad \underline{\omega} = \underline{i}p + \underline{j}q + \underline{k}r = \underline{i}\dot{\varphi} + \underline{j}_1\dot{\theta} + \underline{k}_o\dot{\psi}$$

Using the relations between the unit vectors of Appendix E,

$$(149) \qquad \begin{aligned} p &= \dot{\varphi} - \dot{\psi} \sin \theta \\ q &= \dot{\theta} \cos \varphi + \dot{\psi} \sin \varphi \cos \theta \\ r &= -\dot{\theta} \sin \varphi + \dot{\psi} \cos \varphi \cos \theta \end{aligned}$$

Appendix G

REAC ANALOG OF TORPEDO SYSTEM

The system to be solved on the REAC is described by the system of equations

$$(150)\qquad -m_T V D\beta = Y_\beta \beta + (Y_r - m_L V)\dot{\psi} + Y_{\dot{r}} D\dot{\psi} + Y_{\delta_r}\delta_r$$

$$J_z D\dot{\psi} = N_\beta \beta + N_r \dot{\psi} + N_{\dot{\beta}} D\beta + N_{\delta_r}\delta_r$$

$$(1 + T_\ell D)\delta_r = K_\psi(\psi - \psi_s)$$

where D represents the operation of differentiation with respect to time t. Equations 150 are more readily analoged if they are transformed as follows:

$$(151)\qquad -(m_T V + Y_{\dot{r}}\frac{N_{\dot{\beta}}}{J_z})D\beta = (Y_\beta + Y_{\dot{r}}\frac{N_\beta}{J_z})\beta + (Y_r - m_L V + Y_{\dot{r}}\frac{N_r}{J_z})\dot{\psi}$$

$$+ (Y_{\delta_r} + Y_{\dot{r}}\frac{N_{\delta_r}}{J_z})\delta_r$$

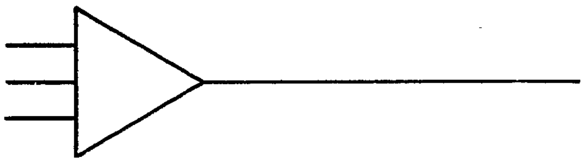
$$(J_z + N_{\dot{\beta}}\frac{Y_{\dot{r}}}{m_T V})D\dot{\psi} = (N_\beta - N_{\dot{\beta}}\frac{Y_\beta}{m_T V})\beta + (N_r - N_{\dot{\beta}}\frac{Y_r - m_L V}{m_T V})\dot{\psi}$$

$$+ (N_{\delta_r} - N_{\dot{\beta}}\frac{Y_{\delta_r}}{m_T V})\delta_r .$$

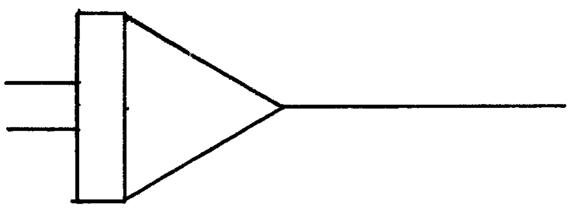
$$D\delta_r = -\frac{1}{T_\ell}\delta_r + \frac{K_\psi}{T_\ell}\psi - \frac{K_\psi}{T_\ell}\psi_s$$

$$\dot{\psi} = D\psi$$

The REAC analog of this system is shown in Fig. 3. The symbols used have the following meaning:

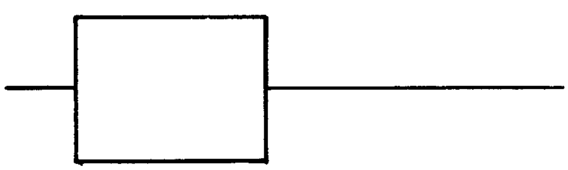


= Summing Amplifier



= Integrating Amplifier

A_{ij} = Gain of amplifier number i with input number j



= Voltage Divider (Potentiometer)

K_i = Setting of potentiometer number i

The amplifier inputs have available gains of 1, 4, and 10. The output is inverted in sign.

The equations of the analog are

(152)
$$e_1 = A_{1.3}K_1 \frac{e_1}{D} + A_{1.2}A_6K_2 \frac{e_2}{D} - A_{1.1}K_3 \frac{e_3}{D}$$

$$e_2 = -A_{2.3}K_4 \frac{e_2}{D} + A_{2.1}A_5K_5 \frac{e_1}{D} - A_{2.2}K_6 \frac{e_3}{D}$$

$$e_3 = -A_{3.3}K_7 \frac{e_3}{D} + A_{3.1}A_4K_8 \frac{e_1}{D^{-2}} - A_{3.2} \frac{e_4}{D}$$

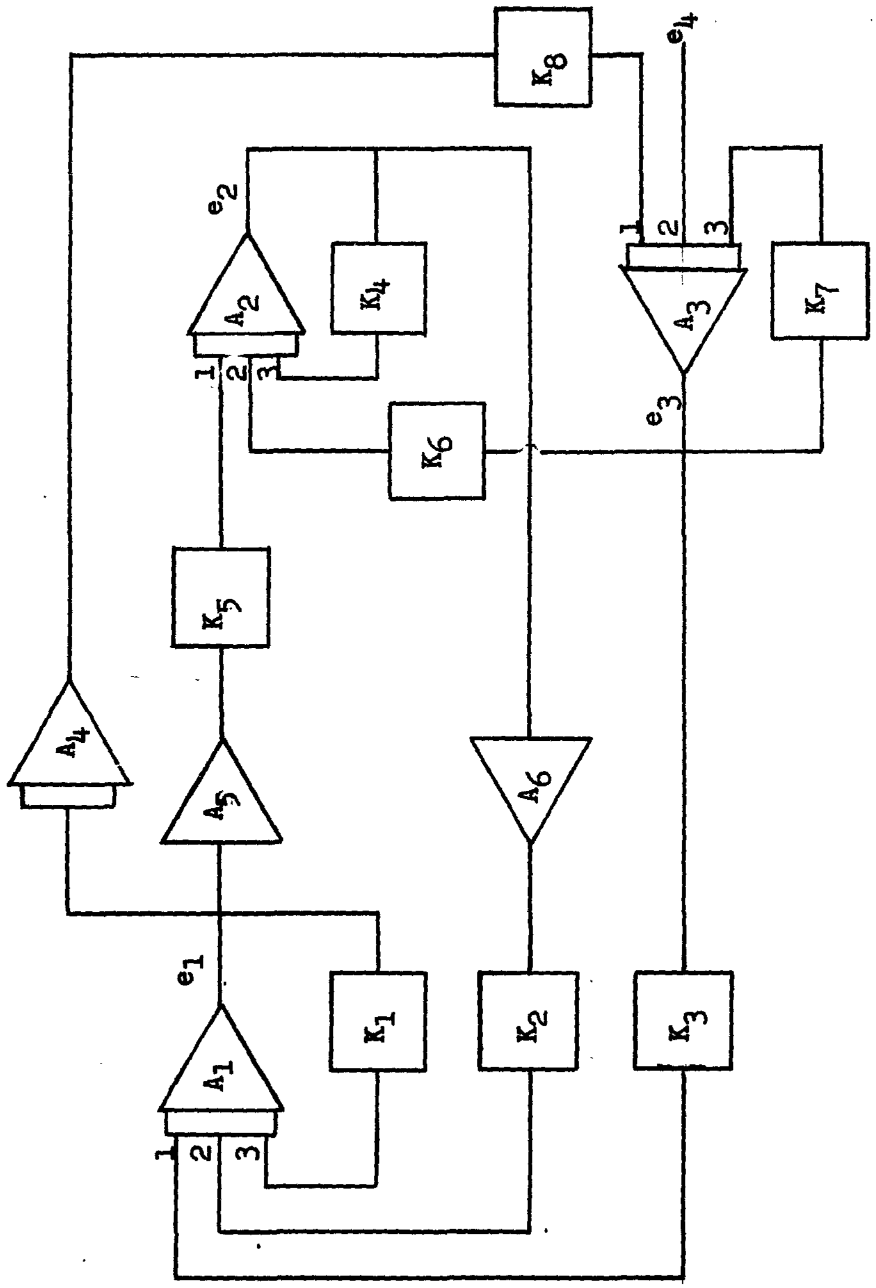


FIG. 3. Analog of Torpedo Yaw Control System.

where $\bar{D}$ is the operator representing differentiation with respect to the computer time base $\bar{t}$. Let $\bar{t} = nt$. Then $D = n\bar{D}$. The voltages e_1, e_2, e_3 are the electrical analogs of the variables $\dot{\psi}$, β, δ_r, respectively, of the physical system. Let

$$a_1e_1 = \dot{\psi} \tag{153}$$

$$a_2e_2 = \beta$$

$$a_3e_3 = \delta_r$$

$$a_4e_4 = \psi_s$$

Equations 152 may then be written

$$D\dot{\psi} = -nA_{1.3}K_1\dot{\psi} + \frac{a_1}{a_2} nA_{1.2}A_6K_2\beta - \frac{a_1}{a_3} nA_{1.1}K_3\delta_r \tag{154}$$

$$D\beta = -nA_{2.3}K_4\beta + \frac{a_2}{a_1} nA_{2.1}A_5K_5\dot{\psi} - \frac{a_2}{a_3} nA_{2.2}K_6\delta_r$$

$$D\delta_r = -nA_{3.3}K_7\delta_r + \frac{a_3}{a_1} n^2A_{3.1}A_4K_8\dot{\psi} - \frac{a_3}{a_4} nA_{3.2}\psi_s$$

By comparing Eq. 154 with Eq. 151 it is seen that the two systems are equivalent if

$$\frac{a_1}{a_2} nA_{1.2}A_6K_2 = + \frac{N_\beta - N_{\dot{\beta}}\dfrac{Y_\beta}{m_TV}}{J_z + N_{\dot{\beta}}\dfrac{Y_{\dot{r}}}{m_TV}}$$

$$nA_{1.3}K_1 = - \frac{N_r - N_{\dot{\beta}}\dfrac{Y_r - m_LV}{m_TV}}{J_z + N_{\dot{\beta}}\dfrac{Y_{\dot{r}}}{m_TV}}$$

$$\frac{a_1}{a_3} nA_{1.1}K_3 = -\frac{N_{\delta_r} - N_{\dot{\beta}}\dfrac{Y_{\delta_r}}{m_T V}}{J_z + N_{\dot{\beta}}\dfrac{Y_{\dot{r}}}{m_T V}}$$

$$nA_{2.3}K_4 = \frac{Y_\beta + Y_{\dot{r}}\dfrac{N_\beta}{J_z}}{m_T V + Y_{\dot{r}}\dfrac{N_{\dot{\beta}}}{J_z}}$$

$$\frac{a_2}{a_1} nA_{2.1}A_5K_5 = -\frac{Y_r - m_L V + Y_{\dot{r}}\dfrac{N_r}{J_z}}{m_T V + Y_{\dot{r}}\dfrac{N_{\dot{\beta}}}{J_z}}$$

$$\frac{a_2}{a_3} nA_{2.2}K_6 = \frac{Y_{\delta_r} + Y_{\dot{r}}\dfrac{N_{\delta_r}}{J_z}}{m_T V + Y_{\dot{r}}\dfrac{N_{\dot{\beta}}}{J_z}}$$

$$nA_{3.3}K_7 = \frac{1}{T_\ell}$$

$$\frac{a_3}{a_1} n^2A_{3.1}A_4K_8 = \frac{K_\psi}{T_\ell}$$

$$\frac{a_3}{a_4} nA_{3.2} = \frac{K_\psi}{T_\ell}$$

Amplifier gains, potentiometer settings, time base change n, and the ratios a_1/a_2, a_1/a_3, a_1/a_4, are chosen in such a manner that the above equations are satisfied. The response to an initial yaw error, obtained by placing the appropriate initial condition on amplifier A_4, may be found.

REFERENCES

1. Davidson, Kenneth S. M. and Leonard I. Schiff, "Turning and Course-Keeping Qualities," SOC NAVAL ARCH MARINE ENGR, TRANS. 1946.

2. Minorsky, N. "Directional Stability of Automatically Steered Bodies," AM SOC NAVAL ENGR, J, Vol. 34 (1922).

3. Minorsky, N. "Control Problems," FRANKLIN INST, J, Vol.232, Nos. 5 and 6 (1941).

4. U. S. Naval Ordnance Test Station, Inyokern. An Investigation of Torpedo Dynamics, by F. J. Bednarz and V. R. Harger, Jr. China Lake, Calif., NOTS, 20 July 1950. (NAVORD Report 1238, NOTS 300), UNCLASSIFIED.

5. ------. Hydrodynamic Simulator, by Firth Pierce and L. W. Sepmeyer. China Lake, Calif., NOTS, 9 February 1949. (NAVORD Report 1142, NOTS 194), CONFIDENTIAL.

6. Page, Leigh. Introduction to Theoretical Physics, New York, Van Nostrand, 1935.

7. California Institute of Technology. Aerodynamic Theory, by W. F. Durand. Durand Reprinting Committee, Pasadena, Calif., CIT, 1943. (Vol. III Div. G.)

8. Lamb, Sir Horace. Hydrodynamics, New York City, Dover Publications, 1945.

9. Brown, Gordon S., and Donald F. Campbell, Principles of Servomechanisms, New York, Wiley, 1948.

10. RAND Corporation. RAND REAC Manual, by A. S. Mengel and W. S. Melahn. Santa Monica, 1950.

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