A NEW TEST OF COMPOUND SYMMETRY

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Institute of Statistics
Mimeograph Series No. 97

March 11, 1954

1 Work sponsored by the Office of Naval Research under Contract No. N7-onr-28402.
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1. **Summary.** If \( x_1 \) and \( x_2 \) have a bivariate normal distribution with a correlation coefficient \( \rho \) and the same standard deviations \( \sigma \) for both, then it is well known and easy to check that \( x_1 + x_2 \) and \( x_1 - x_2 \) are uncorrelated, which forms the basis of Pitman's well-known test of \( H_0: \sigma_1 = \sigma_2 \), for a bivariate normal population, in terms of the correlation coefficient \( r \) between \( x_1 + x_2 \) and \( x_1 - x_2 \) in a random sample of size, say, \( n \), from this population. Starting from this test which has a number of reasonably good properties, and then using the union-intersection principle \( \bigcup_{i=1}^{2} \), a test is obtained for compound symmetry, i.e., for \( H_0: \sigma_{11} = \sigma_{22} = \ldots = \sigma_{pp} \) and all \( \sigma_{ij} \)'s are equal (\( i \neq j = 1, 2, \ldots, p \)), where \( \sigma_{ij} \) is any element of the covariance matrix \( \Sigma \) of a \( p \)-variate normal population.

2. **Test Construction.** Let \( p(x_1, x_2) \) denote the population correlation coefficient between \( x_1 \) and \( x_2 \) and \( r(x_1, x_2) \) the same in a random sample of size, say \( n \), from that population. Then, for a set of stochastic variables \( x'(1 \times p) = (x_1, \ldots, x_p) \) having a \( p \)-variate normal distribution, notice that for any arbitrary non-null vector \( a'(1 \times p) \), \( a'(1 \times p) x(p \times 1) \) and \( \bigcup_{i=1}^{p} x_i \) have a bivariate normal distribution and, now letting \( \sum_{i=1}^{p} a_i = 0 \),
consider the hypothesis: \( p(\sum_{i=1}^{p} x_i, a'x) = 0 = H_{0a} \) (say). Next notice that

\[(2.1) \quad H_0: \text{all } \sigma_{ii}'s \text{ are equal and all } \sigma_{ij}'s \text{ are equal } (i \neq j = 1, \ldots, p) \]

\[= \bigcap_{a} H_{0a} = \bigcap_{a} \left\{ \sum_{i=1}^{p} \rho \left( \sum_{i=1}^{p} x_i, a'x \right) = 0 \right\}, \]

where \( a'(1 \times p) \) is any arbitrary row vector subject to \( \sum_{i=1}^{p} a_i = 0 \).

Now going back to \( H_{0a} \), we have for this hypothesis the Pitman critical region, say \( W(\alpha) \), of size \( \alpha \), given by

\[(2.2) \quad W(\alpha): \quad r^{2}(\sum_{i=1}^{p} x_i, a'x) \geq r_{\alpha}^{2}(n-2), \]

where \( r_{\alpha}(n-2) \) is the upper \( \alpha/2 \)-point of the central \( r \)-distribution in random samples of size \( n \).

Hence, by the union-intersection heuristic principle \( \bigcap_{1, 2} \)
we have, for \( H_0(= \bigcap_{a} H_{0a}) \), the critical region \( W(\beta) \) of size \( \beta \) given by

\[(2.3) \quad W(\beta): \quad \bigcap_{a} \left\{ \sum_{i=1}^{p} \rho^{2}(\sum_{i=1}^{p} x_i, a'x) \geq r_{\beta}^{2}(n-2) \right\}, \]

i.e., \( \text{Sup}_{a} r^{2}(\sum_{i=1}^{p} x_i, a'x) \geq r_{\beta}^{2}(n-2) \)

i.e., \( \text{Sup}_{a} r^{2}(\sum_{i=1}^{p} x_i, \sum_{i=1}^{p} a_i(x_i - x_p)) \geq r_{\beta}^{2}(n-2) \),

remembering that \( \sum_{i=1}^{p} a_i = 0 \), i.e., \( a = - \sum_{i=1}^{p} a_i \).
It is easy to check that

\[ (2.4) \quad \sup_a r^2 \left( \sum_{i=1}^{p} x_i, \sum_{i=1}^{p-1} a_i (x_i - x_p) \right) \]

is square of the sample multiple correlation between \( \sum_{i=1}^{p} x_i \) and the \((p - 1)\)-set of variables \((x_1 - x_p), (x_2 - x_p), \ldots, (x_{p-1} - x_p) =\)

\[ R^2 \sum_{i=1}^{p} x_i \text{ and } (x_1 - x_p), (x_2 - x_p), \ldots, (x_{p-1} - x_p) \]

Notice that, since the new \(p\)-set also has the \(p\)-variate normal distribution, this \(R\) has the well-known multiple correlation distribution with degrees of freedom \(p - 1\) and \(n - p\) and a non-centrality parameter which is the population multiple correlation between \( \sum_{i=1}^{p} x_i \) and the \((p - 1)\)-set above and which let us call \( \rho^2 \). It is easy to check that \( \rho = 0 \), i.e., that \( R \) has the central multiple correlation distribution, if and only if

\[ \rho(\sum_{i=1}^{p} x_i, x_i - x_p) = 0 \quad (i = 1, 2, \ldots, p - 1), \text{ i.e., if and only if} \]

\[ \rho(\sum_{i=1}^{p} x_i, a'x) = 0 \quad (\text{for all non-null } a \text{ subject to } \sum_{i=1}^{p} a_i = 0). \]

We have thus, for testing compound symmetry, the critical region \( W(\beta) \) of size \( \beta \) given by

\[ (2.5) \quad R \left( \sum_{i=1}^{p} x_i \text{ and } (x_1 - x_p), \ldots, (x_{p-1} - x_p) \right) \leq \beta(p - 1, n - p), \]

when \( R_{\beta} \) is the upper \( \beta \)-point of the well-known central multiple correlation distribution.
It can be checked after some little algebra that in terms, respectively, of the elements of the sample and population covariance matrices $S$ and $\Sigma$, $R$ and $\rho$ will be given by

\[ R^2 = 1 - p \sum_{i=1}^{p} \frac{z_i^2}{\sum_{i=1}^{p} z_i^2}, \]

\[ \rho^2 = 1 - p \sum_{i=1}^{p} \frac{\zeta_i^2}{\sum_{i=1}^{p} \zeta_i^2}, \]

when

\[ z_i = \sum_{j=1}^{P} s_{ij}, \quad z_i^1 = \sum_{j=1}^{P} s_{i1}^1, \]

\[ \zeta_i = \sum_{j=1}^{P} \sigma_{ij}, \quad \zeta_i^1 = \sum_{j=1}^{P} \sigma_{i1}^1. \]

Note that $s_{ij} = s_{ji}$, $s_{i1}^1 = s_{1i}^1$, $\sigma_{ij} = \sigma_{ji}$ and $\sigma_{i1}^1 = \sigma_{1i}^1$.

The power properties of this test will be discussed in a later note.

References
