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[illegible]RIC THEORY OF THE DIFFERENTIAL EQUATIONS OF EXTERIOR BALLISTICS

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U. S. NAVAL ORDNANCE LABORATORY

WHITE OAK, MARYLAND

METRIC THEORY OF THE DIFFERENTIAL EQUATIONS OF EXTERIOR BALLISTICS

Prepared by:

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ABSTRACT: This paper develops the fundamental equations of motion for an axially symmetric shell using the so-called "spinless" reference frame, suggested by A. W. Wundheiler. Approximate integration methods are also considered, not so much for the purpose of solving the equations, as to give the proper background for the application of the author's "metric theory" to this particular system. A particular Riemannian metric is introduced and, to some extent, justified as being particularly efficient for this purpose. The theory requires the calculation of the coefficients of a certain quadratic form in eight independent variables. Explicit formulas are given for the most complicated of these coefficients. They are indeed so complex that the author has not succeeded in drawing interesting general conclusions from them. They are nevertheless available for the numerical analysis of particular cases.

U. S. NAVAL ORDNANCE LABORATORY
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This report represents a new approach to the linearized equations of exterior ballistics. The novel features include use of a "spinless" reference frame and the application of the authors "metric theory" of differential equations. The work was carried out under project NOL-Re9a-108-1-53, entitled "Aerodynamics and Fluid Mechanics." The results are of use for the solution of exterior ballistics problems.

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1. Introduction. In two previous papers,[1] hereafter denoted by MP_1 and MP_2, the author has developed a "metric theory" of ordinary differential equations with the view of studying the dependence of the solutions on the initial conditions. In Section 4 of MP_2 some indications were given as to how the theory could be applied to the equations of exterior ballistics. The purpose of this paper is to go into more details with regard to this application. The ultimate goal is to obtain results suitable for numerical analysis as well as for a clear qualitative picture of what can be expected in problems of this sort.

2. The equations of motion of a spinning symmetric shell. We use a moving cartesian reference frame F with origin 0 at the center of gravity of the shell and with one of its axes (namely the x_0-axis) coinciding with the shell axis and directed from 0 to the shell vertex. The other two axes (namely the x_1-axis and x_2-axis) are chosen in such a way that $0\text{-}x_1x_2x_0$ form a right handed system. It is not necessary to assume that the shell is rigidly attached to this reference frame F. In fact we simplify our equations slightly if we assume, as we do from now on, that the component of the angular velocity of F in the direction Ox_0 is zero,[2] while that of the shell is ω_0, in general $\neq 0$. On the other hand the components of angular velocity of both F and the shell in the direction of the other two axes are the same and will be denoted by ω_1 and ω_2 respectively. Hence, if Ω_F is the vector angular velocity of F (relative to some fixed frame F_0) and if Ω_S is the vector angular velocity of the shell, we have

$$\Omega_S = \omega_0 i + \omega_1 j + \omega_2 k$$

$$\Omega_F = 0i + \omega_1 j + \omega_2 k,$$

where i, j, k are unit vectors directed along the x_0, x_1, x_2 axes respectively.

Similarly we denote the vector velocity V of the point 0(relative to the fixed frame F_0) in the form

$$V = u_0 i + u_1 j + u_2 k,$$

while the vector G representing the acceleration of gravity and having constant components relative to the fixed frame F_0 appears in the form

$$G = g_0 i + g_1 j + g_2 k.$$

Denoting the mass of the shell by m, its axial moment of inertia by a, and its transverse moment of inertia by b, we can write the moment of momentum in the form,

$$\mathcal{M} = a\,\omega_0 i + b\,\omega_1 j + b\,\omega_2 k$$

and the momentum appears in the form mV. We denote the vector aerodynamic moment by

$$M = aM_0 i + bM_1 j + bM_2 k,$$

the factors a and b being introduced for later convenience; and we denote vector aerodynamic force by

$$m\,\overline{\Phi} = m\,\Phi_0 i + m\,\Phi_1 j + m\,\Phi_2 k.$$

We now are in a position to write down three vector differential equations. The first of these is

$$(2.1) \qquad m\dot{V} \equiv m\frac{dV}{dt} + m\,\Omega_F \times V = mG + m\,\overline{\Phi},$$

which equates the rate of change of momentum (relative to the fixed frame F_0) to the resultant of the applied forces. The second of our equations is

$$(2.2) \qquad \dot{\mathcal{M}} \equiv \frac{d\mathcal{M}}{dt} + \Omega_F \times \mathcal{M} = M,$$

which equates the rate of change of moment of momentum to the applied torque. Finally, the third equation

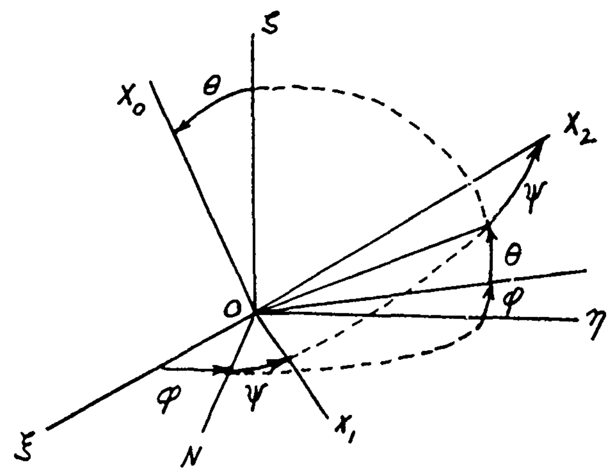
$$(2.3) \qquad \dot{G} \equiv \frac{dG}{dt} + \Omega_F \times G = 0$$

merely expresses the fact that G is a constant vector; that is, it has constant components relative to the fixed frame F_0.

Now $\overline{\Phi}$ and M are assumed to be functions of Ω_S and V only. The precise nature of this dependence will be specified in detail later on. We only wish to remark now that the above three vector equations become self contained. They are equivalent to nine scalar differential equations for the determination of nine unknown functions, namely $\omega_0, \omega_1, \omega_2$, u_0, u_1, u_2, g_0, g_1, g_2. Actually the order of the system can be reduced from nine to eight, because of the obvious first integral $G \cdot G = \text{const}$. When once these unknown functions have been determined by integration of (2.1), (2.2) and (2.3), the trajectory may be calculated as follows:

We introduce a moving frame F_0' with the same origin as F (namely the center of gravity of the shell) but with its axes invariably parallel to the corresponding axes of the fixed frame F_0. Eulerian angles θ, φ, ψ are now introduced to define the orientation of F relative to F_0' and hence also relative to F_0. We hereby assume that the axes $O\zeta$, $O\xi$, $O\eta$ of F_0' form a right handed system, that θ is the angle from $O\zeta$ to Ox_0, that φ is an angle from $O\xi$ to the intersection ON of the $\xi\eta$-plane with the x_1x_2-plane (the so-called line of nodes), and that ψ is an angle from this same line of nodes ON to the Ox_1 axis. Cf. the figure.

We further assume that F_0 and hence F_0' is chosen in such a way that one of its axes, say the ζ-axis, points vertically upward. Then, since gravity always acts vertically downward we see easily that,



$$g_0 = -|G| \cos\theta$$

$$g_1 = -|G| \sin\theta \sin\psi$$

$$g_2 = -|G| \sin\theta \cos\psi$$

from which we find that $\tan\psi = g_1/g_2$ and $\tan\theta = \pm\sqrt{g_1^2 + g_2^2}/g_0$. When ψ and θ are thus found as functions of t (since g_0, g_1, g_2 are known functions of t after the integration of (2.1), (2.2), and (2.3)), we can find φ by the single quadrature involved in integrating the equation

$$\frac{d\psi}{dt} + \frac{d\varphi}{dt} \cos\theta = 0.$$

This equation expresses the fact that the frame F has zero angular velocity about Ox_0. Thus

$$\varphi = \varphi_0 - \int_0^t \dot{\psi} \sec\theta \; dt. \tag{2.4}$$

With the Eulerian angle all determined, we can now find the nine direction cosines between the frames F and F_0' and hence between F and F_0. They are exhibited in the following table,

	X	Y	Z
x_1	$\lambda_1=\cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta$	$\lambda_2=\sin\varphi\cos\psi + \cos\varphi\sin\psi\cos\theta$	$\lambda_3=\sin\psi\sin\theta$
x_2	$\mu_1=-\cos\varphi\sin\psi - \sin\varphi\cos\psi\cos\theta$	$\mu_2=-\sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\theta$	$\mu_3=\cos\psi\sin\theta$
x_0	$\nu_1=\sin\theta\sin\varphi$	$\nu_2=-\sin\theta\cos\varphi$	$\nu_3=\cos\theta$

where the axes of the fixed frame F_0 are denoted by X, Y, Z. This table may easily be obtained from the law of cosines in spherical trigonometry or by the matrix multiplication corresponding to the fact that F may be obtained from F_θ' by rotating the latter through φ about the ζ axis, then through θ about the carried ξ axis, and finally through ψ about the carried ζ -axis.

The components $\frac{dX}{dt}$, $\frac{dY}{dt}$ and $\frac{dZ}{dt}$ of the shell velocity relative to F_0 are, of course, related to the components u_1, u_2, u_0 relative to F by means of the following equations:

$$\frac{dX}{dt} = \lambda_1 u_1 + \mu_1 u_2 + \nu_1 u_0$$

$$\frac{dY}{dt} = \lambda_2 u_1 + \mu_2 u_2 + \nu_2 u_0 \tag{2.5}$$

$$\frac{dZ}{dt} = \lambda_3 u_1 + \mu_3 u_2 + \nu_3 u_0.$$

Since the direction cosines denoted by the λ's, μ's, and ν's are known functions of t as well as the u's (when once equations (2.1), (2.2), (2.3), and (2.4) have been integrated), the actual trajectory may be obtained by the three obvious quadratures presented in (2.5).

Thus the key to the entire situation is the integration of the self contained system of ninth order represented by the three vector equations (2.1), (2.2), and (2.3). It is this system to which we shall apply the metric theory of our previous papers. Written out in full, these nine equations take the following form

(2.6) $$\frac{du_0}{dt} + \omega_1 u_2 - \omega_2 u_1 = g_0 + \Phi_0$$

(2.7) $$\frac{du_1}{dt} + \omega_2 u_0 = g_1 + \Phi_1$$

(2.8) $$\frac{du_2}{dt} - \omega_1 u_0 = g_2 + \Phi_2$$

(2.9) $$\frac{d\omega_0}{dt} = M_0$$

(2.10) $$\frac{d\omega_1}{dt} + h\,\omega_0\,\omega_2 = M_1 \quad \text{(where } h = a/b\text{)}$$

(2.11) $$\frac{d\omega_2}{dt} - h\omega_0\,\omega_1 = M_2$$

(2.12) $$\frac{dg_0}{dt} + \omega_1 g_2 - \omega_2 g_1 = 0$$

(2.13) $$\frac{dg_1}{dt} + \omega_2 g_0 = 0$$

(2.14) $$\frac{dg_2}{dt} - \omega_1 g_0 = 0.$$

If we introduce complex quantities defined as follows, the number of our equations can be reduced from nine to six. Since three of these six equations are complex, this simplication is formal rather than actual. Still it is well worth carrying out. We set

$$\omega = \omega_1 + i\,\omega_2 \qquad (i = \sqrt{-1})$$

$$u = u_1 + iu_2$$

(2.15) $$\Phi = \Phi_1 + i\Phi_2 \quad \text{(not to be confused with vector } \bar{\Phi}\text{)}$$

$$M = M_1 + iM_2$$

$$g = g_1 + ig_2$$

Then multiplying (2.8) by i and adding to (2.7) we obtain, with the help of (2.15), the following complex equation

(2.16) $$\frac{du}{dt} = iu_0\omega + g + \Phi$$

to be used as the equivalent of the two equations (2.7) and (2.8). In a similar manner (2.10) and (2.11) are replaced by

(2.17) $$\frac{d\omega}{dt} = ih\,\omega_0\omega + M,$$

while (2.13) and (2.14) are replaced by

(2.18) $$\frac{dg}{dt} = ig_0\omega$$

3. Expressions for the aerodynamic force and torque. The use of complex quantities also simplifies the discussion of the effect of axial symmetry[3] on Φ and M. Thus $M = M(\omega_0, \omega, u_0, u)$. A rotation of the frame F through an angle α about Ox_0 replaces M by $Me^{i\alpha}$, ω by $\omega e^{i\alpha}$, and u by $ue^{i\alpha}$. Hence, by axial symmetry, we must have

(3.1) $$M(\omega_0, \omega, u_0, u)e^{i\alpha} = M(\omega_0, \omega e^{i\alpha}, u_0, ue^{i\alpha})$$

as a functional equation to be satisfied by M for all real values of α. If we consider only terms that are linear in the relatively small quantities u_1, u_2, ω_1, and ω_2, it is readily shown from (3.1) that the most general form for M would be

(3.2) $$M = A\omega + Bu$$

where A and B are complex valued functions of the real quantities u_0 and ω_0[4]. A similar argument shows that

(3.3) $$\Phi = C\omega + Du,$$

where C and D are also complex valued functions of u_0 and ω_0, and that

(3.4) $$M_0 = M_0\,(\omega_0, u_0)$$

(3.5) $$\Phi_0 = \Phi_0\,(\omega_0, u_0)$$

where M_0 and Φ_0 are real valued functions of u_0 and ω_0.

If we assume that A can be developed in powers of u_0 and ω_0 and if we then neglect all terms of higher degree than the first, we have

$$A = (a_1 + a_{11}\omega_0 + a_{12}u_0) + i(a_2 + a_{21}\omega_0 + a_{22}u_0).$$

Thus there are six real "aerodynamic constants" associated with A, namely a_1, a_2, a_{11}, a_{12}, a_{21}, a_{22}. Similarly we write

$$B = (b_1 + b_{11}\omega_0 + b_{12}u_0) + i(b_2 + b_{21}\omega_0 + b_{22}u_0)$$

$$C = (c_1 + c_{11}\omega_0 + c_{12}u_0) + i(c_2 + c_{21}\omega_0 + c_{22}u_0)$$

$$D = (d_1 + d_{11}\omega_0 + d_{12}u_0) + i(d_2 + d_{21}\omega_0 + d_{22}u_0)$$

$$M_0 = m_0 + m_1\omega_0 + m_2u_0 + m_{11}\omega_0^2 + m_{12}\omega_0 u_0 + m_{22}u_0^2$$

$$\Phi_0 = f_0 + f_1\omega_0 + f_2u_0 + f_{11}\omega_0^2 + f_{12}\omega_0 u_0 + f_{22}u_0^2$$

Thus we consider at the outset a total of 36 aerodynamic constants. However, it is physically clear that $M_0 = 0$ if $\omega_0 = 0$. Hence $m_0 = m_2 = m_{22} = 0$. Also $\Phi_0 = 0$ if $u_0 = 0$. Hence $f_0 = f_1 = f_{11} = 0$. Also M_0 is an odd function of ω_0 while Φ_0 is an even function of ω_0. Hence $m_{11} = 0$ and $f_{12} = 0$. The number of constants is thus reduced at once to 28. Furthermore, it seems to be generally agreed that of these 28 constants a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, m_1, and f_2 may be neglected. We are thus left with 18 constants and the above equations reduce to

$$A = (a_{11}\omega_0 + a_{12}u_0) + i(a_{21}\omega_0 + a_{22}u_0)$$

$$B = (b_{11}\omega_0 + b_{12}u_0) + i(b_{21}\omega_0 + b_{22}u_0)$$

$$C = (c_{11}\omega_0 + c_{12}u_0) + i(c_{21}\omega_0 + c_{22}u_0)$$

$$D = (d_{11}\omega_0 + d_{12}u_0) + i(d_{21}\omega_0 + d_{22}u_0)$$

$$M_0 = m_{12}\omega_0 u_0$$

$$\Phi_0 = f_{22}u_0^2$$

The elimination of eight more constants can be accomplished as follows: If $\omega_1 = u_2 = 0$ equations (3.2) and (3.3) become $M = Ai\,\omega_2 + Bu_1$, $\Phi = Ci\,\omega_2 + Du_1$; so that, upon separating real and pure imaginary parts, we have (with the help of the above expressions for A, B, C, D) the following relations:

$$M_1 = -(a_{21}\omega_0 + a_{22}u_0)\omega_2 + (b_{11}\omega_0 + b_{12}u_0)u_1$$

$$M_2 = (a_{11}\omega_0 + a_{12}u_0)\omega_2 + (b_{21}\omega_0 + b_{22}u_0)u_1$$

$$\Phi_1 = -(c_{21}\omega_0 + c_{22}u_0)\omega_2 + (d_{11}\omega_0 + d_{12}u_0)u_1$$

$$\Phi_2 = (c_{11}\omega_0 + c_{12}u_0)\omega_2 + (d_{21}\omega_0 + d_{22}u_0)u_1.$$

But, with $\omega_1 = u_2 = 0$, the vector velocity of every point on the shell axis must be orthogonal to the x_2-axis. Hence the component in the direction of the x_1-axis of any element of aerodynamic force acting on the shell would not change sign if the direction of spin were reversed, but the component in the direction of the x_2-axis would change its sign under such circumstances. In reaching this conclusion we use considerations of symmetry in the form sometimes called "the principle of sufficient reason" (cf. G. D. Birkhoff, Collected Mathematical Papers, volume 3, pp. 778-804). We thus conclude that, with $\omega_1 = u_2 = 0$, Φ_1 and M_2 must be even functions of ω_0, while Φ_2 and M_1 must be odd functions of ω_0. We thus read off from the above equations that $a_{22} = b_{12} = a_{11} = b_{21} = c_{21} = d_{11} = c_{12} = d_{22} = 0$. Hence our final formulas for A, B, C, D, M_0, and Φ_0 are as follows:

$$A = a_{12}u_0 + ia_{21}\omega_0$$

$$B = b_{11}\omega_0 + ib_{22}u_0$$

$$C = c_{11}\omega_0 + ic_{22}u_0$$

$$D = d_{12}u_0 + id_{21}\omega_0$$

$$M_0 = m_{12}\omega_0 u_0$$

$$\Phi_0 = f_{22}u_0^{\,2}$$

The ten remaining constants are usually expressed in terms of dimensionless quantities which are generally denoted by the capital letter K with various subscripts. In accordance with one well known usage we write

$$a_{12} = -\rho d^4 K_H b^{-1} \qquad a_{21} = \rho d^5 K_{XT} b^{-1}$$

$$b_{11} = -\rho d^4 K_T b^{-1} \qquad b_{22} = -\rho d^3 K_M b^{-1}$$

$$c_{11} = \rho d^4 K_{XF} m^{-1} \qquad c_{22} = \rho d^3 K_S m^{-1}$$

$$d_{12} = -\rho d^2 K_N m^{-1} \qquad d_{21} = \rho d^3 K_F m^{-1}$$

$$m_{12} = -\rho d^4 K_A a^{-1} \qquad f_{22} = -\rho d^2 K_{DA} m^{-1}$$

Here ρ = density of air (= .001188 gm/cm^3 for a typical value)
d = diameter of projectile (2 centimeters in a typical case)
m = mass of projectile (175 grams)
b = transverse moment of inertia (800 gm - cm^2)
a = axial moment of inertia (76 gm - cm^2)

According to some experimental work of Turetsky, the dimensionless aerodynamic coefficients may be expected to have values of about the following magnitude: $K_H = 6$, $K_T = -0.1$, $K_M = 1$, $K_S = -10$. $K_N = 1$, $K_F = 0.2$. $K_A = 0.005$, $K_{DA} = 0.1$. I have no information about K_{XT} and K_{XF}. They are probably too small to be significant.

Needless to say, the aerodynamic force system described here does not satisfy the Nielsen-Synge requirement of invariance with respect to shift of mass center.

4. Digression on the approximate integration of linear differential equations. The approximate solution of the differential system introduced above hinges on the solution of a system of linear equations of the following form:

$$\frac{dw}{dt} = \alpha(t)w + \beta(t)u + H(t)$$

(4.1)

$$\frac{du}{dt} = \gamma(t)w + \delta(t)u + L(t)$$

where w, u, α, β, γ, δ, H, and L are complex valued functions of the real variable t. The solution of such a system of non-homogeneous linear differential equations is well known to depend on the solution of the corresponding homogeneous system, in which H(t) and L(t) are both zero. But it is possible to write down immediately an approximate solution of the homogeneous system, at least if $h^2 = \frac{1}{4}\left[(\alpha - \delta)^2 + 4\beta\gamma\right]$ and β are not zero and the derivatives of α, β, γ, δ are small enough compared to h and β. In fact the functions

$$
\begin{aligned}
W &= \beta^{\frac{1}{2}} h^{-\frac{1}{2}} \exp \int \left[\tfrac{1}{2}(\alpha + \delta) + h\right] dt \\
U &= \left[h - \tfrac{1}{2}(\alpha - \delta)\right] \beta^{-\frac{1}{2}} h^{-\frac{1}{2}} \exp \int \left[\tfrac{1}{2}(\alpha + \delta) + h\right] dt
\end{aligned} \tag{4.2}
$$

may be verified to satisfy exactly the equations

$$
\begin{aligned}
\frac{dW}{dt} &= \left(\alpha + \frac{1}{2}\frac{\dot\beta}{\beta} - \frac{1}{2}\frac{\dot h}{h}\right)W + \beta U \\
\frac{dU}{dt} &= \left(\gamma + \frac{1}{2}\frac{\dot h}{h}\,\frac{\alpha - \delta}{\beta} - \frac{1}{2}\,\frac{\dot\alpha - \dot\delta}{\beta}\right)W + \left(\delta + \frac{1}{2}\frac{\dot h}{h} - \frac{1}{2}\frac{\dot\beta}{\beta}\right)U,
\end{aligned} \tag{4.3}
$$

which, in view of the smallness of $\dot h$, $\dot\beta$ and $\dot\alpha - \dot\delta$, are only slight modifications of (4.1) when H(t) = L(t) = 0. Moreover from the definition of h as a square root of a non-vanishing quantity, it is clear that (4.2) yields two solutions, one for each determination of the sign of the square root. The verification that (4.2) satisfies (4.3) is laborious but elementary and will be left to the reader.

To obtain a solution of the non-homogeneous linear system (4.1) we need to know two particular solutions of the homogeneous system depending on a real parameter s, namely $w_1(t,s)$, $u_1(t,s)$ and $w_2(t,s)$, $u_2(t,s)$, such that $w_1(s,s) = 1$, $u_1(s,s) = 0$, $w_2(s,s) = 0$, $u_2(s,s) = 1$. Then, as is well known and as is easily verified, it turns out that a particular solution of (4.1) is given by

$$
\begin{aligned}
w &= \int_0^t \left[H(s)\, w_1(t,s) + L(s)\, w_2(t,s)\right] ds \\
u &= \int_0^t \left[H(s) u_1(t,s) + L(s) u_2(t,s)\right] ds
\end{aligned} \tag{4.4}
$$

But from the approximate solution (4.2) of the homogeneous equations it is easy approximately to set up these functions $w_1(t,s)$ and $u_1(t,s)$. In fact, using the abbreviations

$$P(t) = \beta^{\frac{1}{2}} h^{-\frac{1}{2}}$$

$$Q_1(t) = (h - \tfrac{1}{2}(\alpha - \delta))\, \beta^{-\frac{1}{2}} h^{-\frac{1}{2}}$$

$$Q_2(t) = (-h - \tfrac{1}{2}(\alpha - \delta))\, \beta^{-\frac{1}{2}} h^{-\frac{1}{2}}$$

we write

$$w_1(t,s) = \tfrac{1}{2}P(t)\left[Q_1(s) \exp \int_s^t (\tfrac{1}{2}(\alpha+\delta) - h)d\tau - Q_2(s) \exp \int_s^t (\tfrac{1}{2}(\alpha+\delta) + h)d\tau\right]$$

(4.5.1)

$$u_1(t,s) = \tfrac{1}{2}\left[Q_2(t)Q_1(s) \exp \int_s^t (\tfrac{1}{2}(\alpha+\delta) - h)d\tau - Q_1(t)Q_2(s) \exp \int_s^t (\tfrac{1}{2}(\alpha-\delta)+h)d\tau\right]$$

$$w_2(t,s) = \tfrac{1}{2}P(t)P(s)\left[\exp \int_s^t (\tfrac{1}{2}(\alpha+\delta) + h)d\tau - \exp \int_s^t (\tfrac{1}{2}(\alpha+\delta) - h)d\tau\right]$$

(4.5.2)

$$u_2(t,s) = \tfrac{1}{2}P(s)\left[Q_1(t) \exp \int_s^t (\tfrac{1}{2}(\alpha+\delta) + h)d\tau - Q_2(t) \exp \int_s^t (\tfrac{1}{2}(\alpha+\delta) - h)d\tau\right].$$

Both $w_1(t,s)$, $u_1(t,s)$ and $w_2(t,s)$, $u_2(t,s)$, as given by (4.5), satisfy (4.3) exactly, while (4.4), with these values for the w_1 and u_1 inserted, will satisfy exactly the following system:

$$\frac{dw}{dt} = (\alpha + \tfrac{1}{2}\frac{\dot\beta}{\beta} - \tfrac{1}{2}\frac{\dot h}{h})w \qquad + \beta u \qquad + H(t)$$

(4.6)

$$\frac{du}{dt} = (\gamma + \tfrac{1}{2}\frac{\dot h}{h}\;\frac{\alpha - \delta}{\beta} - \tfrac{1}{2}\;\frac{\dot\alpha - \dot\delta}{\beta}) + (\delta + \tfrac{1}{2}\frac{\dot h}{h} - \tfrac{1}{2}\frac{\dot\beta}{\beta})u + L(t),$$

which, of course, is a close approximation of (4.1). Thus the functions given by (4.4) will approximate closely the solution of (4.1) which vanishes when $t = 0$, and estimates of this approximation can be carried out by classical methods.

5. The integration of the equations of motion. We begin by writing down in a suitable form the equations developed in Section 2 and 3. Namely we have equations (2.6), (2.9), (2.18), (2.17), (2.16), while (2.12) is eliminated by setting $g_0 = -(|G|^2 - g_1^2 - g_2^2)^{\frac{1}{2}}$. We also substitute for Φ_0, Φ, M_0, and M their values in terms of the aerodynamic constants discussed in Section 3. The result is the following system of five equations:

$$(5.1) \qquad du_0/dt = \omega_2 u_1 - \omega_1 u_2 - (|G|^2 - |g|^2)^{\frac{1}{2}} + f_{22} u_0^2$$

$$(5.2) \qquad d\omega_0/dt = m_{12}\omega_0 u_0$$

$$(5.3) \qquad dg/dt = -iw(|G|^2 - |g|^2)^{\frac{1}{2}}$$

$$(5.4) \qquad dw/dt = a_{12}u_0 + i(a_{21} + h)\omega_0)w + (b_{11}\omega_0 + ib_{22}u_0)u$$

$$(5.5) \qquad du/dt = (c_{11}\omega_0 + i(c_{22} + 1)u_0)w + (d_{12}u_0 + id_{21}\omega_0)u + g$$

We have set $\omega = w$ in order to conform to the notation of the previous section. Since the last three equations are complex equations, they are equivalent to six real equations. Hence the order of our system (considered as a real system) is eight, even though for some purposes it may be treated as of order five. The quantities $|G|$, f_{22}, m_{12}, a_{12}, a_{21}, h, b_{11}, b_{22}, c_{11}, c_{22}, d_{12}, d_{21} are, of course, all constants, while u_0, ω_0 $g = g_1 + ig_2$, $w = \omega_1 + i\omega_2$, $u = u_1 + iu_2$ are the unknown functions of t.

All five equations can be trivially satisfied by taking g, w, and u identically zero and then by integrating the first two equations by simple quadratures. But this would correspond to shots fired either straight up or straight down and is of no practical significance.

Aside from this there is no known method of obtaining exact elementary solutions of these equations, although the classical existence theorems not only assert that solutions corresponding to arbitrary initial values of the unknowns do exist but give implicitly a means for the numerical calculation of these solutions to any desired degree of accuracy. These methods usually start with a crude approximation (which we call the zero^th^ approximation) and then proceed by successive approximations. The n^{th} approximation, namely, is obtained from the $(n-1)^{th}$ approximation, by substituting the $(n-1)^{th}$ approximation in the right hand members of (5.1) - (5.5) and then integrating under the appropriate initial conditions. It is known (under suitable precautions) that the approximations converge uniformly to the exact solution and, in fact, they converge quite rapidly if the time interval is not unreasonably large, no matter how crude the zeroth approximation may be. We shall, however, focus attention on how to get a really refined zeroth approximation, which is sufficiently accurate so that the first approximation may be considered (at least for some purposes) as the actual solution.

We hereby limit ourselves to initial conditions in which the angle between the shell axis and the tangent to the trajectory is small. Whether this so-called "yaw-angle" remains small depends on the differential equations. For constants corresponding to a properly constructed shell it should remain small, and our zeroth approximation will in fact be obtained by taking it actually equal to zero. It is for such properly constructed shells that our system of successive approximations is to be expected to converge very rapidly. For improperly constructed shells the approximations theoretically would still converge but in a much slower and more unstable manner.

We denote by $\bar{\theta}$ the angle between the tangent to the trajectory and the vertical direction (i.e. the ζ-axis). If the yaw angle is small then $\bar{\theta} - \theta$ is small (although the converse is not necessarily true). In all cases $\bar{\theta} - \theta$ is to be taken as small for $t = 0$, and in the zeroth approximation $\bar{\theta} - \theta = 0$ for all t.

A more complete statement of the initial conditions for both the zeroth approximation and the actual solution is indicated in the following table:

Variable	For the zeroth approximation	For the actual solution
u_0	> 0 (large)	> 0 (large)
u	$= 0$	complex number with small modulus
ω_0	> 0 (large)	> 0 (large)
w	$= 0$	complex number with small modulus
θ	angle between 0 and π*	angle between 0 and π*
g_0	$= - \lvert G \rvert \cos \theta$*	approximately the same
g_2	$= - \lvert G \rvert \sin \theta$*	approximately the same
g_1	$= 0$	small real number
φ	$= 0$	small
ψ	$= 0$	small

*holds also throughout the motion

Taking g_1, φ, and ψ all equal to zero initially means that the shell axis is initially in the YZ (or $\eta\zeta$) plane. If, in addition, $u = 0$, the trajectory is also in this plane, at least, initially.

To get the zeroth approximation we assume that the center of gravity of the shell follows the trajectory of a particle in exterior ballistics, in which the drag is proportional to the square of the velocity, and that the yaw is zero. This amounts to replacing equations (5.1) and (5.3) by

(5.6) $$\frac{du_0}{dt} = -(|G|^2 - g_2{}^2)^{\frac{1}{2}} + f_{22}u_0{}^2$$

and

(5.7) $$\frac{dg_2}{dt} = \frac{g_2\sqrt{|G|^2 - g_2{}^2}}{u_0}, \quad \frac{dg_1}{dt} = 0$$

respectively. Here the drag coefficient is, of course, $-f_{22}$. These equations are obtained from Newton's second law of motion by taking components in the tangential and normal directions; and (5.7) when expressed in terms of $\bar{\theta}$ takes the more familiar form

$$\frac{d\bar{\theta}}{dt} = -\frac{|G| \sin\bar{\theta}}{u_0}$$

(with $\theta = \bar{\theta}$, as previously explained, and $g_2 = -|G| \sin\theta$).

As is well known, (5.6) and (5.7) can be integrated by quadratures in terms of elementary functions. This is effected by eliminating t and using $p = u_0g_2$ as a new variable. Thus we get

$$dp/dg_2 = -f_{22}p^3/g_2{}^3(|G|^2 - g_2{}^2)^{\frac{1}{2}}$$

in which the variables are separable. Of course, the actual integration of (5.6) and (5.7) by quadratures in the indicated manner may present practical difficulties. Approximate solutions are for the present purposes quite good enough.

Having once obtained the zeroth approximation for u_0 as a function of t, the zeroth approximation for ω_0 is obtained by integrating (5.2). When this has been accomplished, so that both u_0 and ω_0 as well as g, or rather their zeroth approximations, are known functions of t, we find that equations

(5.4) and (5.5) take the form (4.1), with $H(t) = 0$ and $L(t) = g$. Good zeroth approximations for w and u can therefore be obtained from formulas (4.4). In making this statement we use the intuitively obvious fact that both u_0 and ω_0 have comparatively small time rates of change. In other words our zeroth approximation is best when du_0/dt and $d\omega_0/dt$ are smallest. This will insure the smallness of the quantities $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\delta}$ of the preceding section.

6. <u>On the most favorable metric for the ballistic equations.</u> The sensitivity of the motion of the shell to slight variations in the initial conditions was the original subject of this research project. This is closely connected, if not identical, with the question of stability. But so far the studies of this question have been limited to systems corresponding to equations (5.4) and (5.5) in their application to the finding of approximations. In such applications, the u_0, ω_0 and g are regarded as known. In fact it is sometimes assumed in effect that they are even constant. The condition for stability, as given by Nielsen and Synge[5] for example, is then to the effect that the determinantal equation

$$(6.1) \qquad \begin{vmatrix} \alpha - \sigma & \beta \\ \gamma & \delta - \sigma \end{vmatrix} = 0$$

in σ should be such that the real part of each root should be negative (or possibly zero). Here, in conformity with our previous notation, we define $\alpha, \beta, \gamma, \delta$ as follows:

$$(6.2) \qquad \begin{aligned} \alpha &= a_{12}u_0 + i(a_{21} + h)\omega_0 \\ \beta &= b_{11}\omega_0 + ib_{22}u_0 \\ \gamma &= c_{11}\omega_0 + i(c_{22} + 1)u_0 \\ \delta &= d_{12}u_0 + id_{21}\omega_0. \end{aligned}$$

so that equations (5.4) and (5.5) appear as (4.1). Such a theory of stability does indeed take into account the part of the mathematical theory which one intuitively feels is the most crucial. Nevertheless it focuses exclusive attention on (5.4) and (5.5) and ignores the fact $\alpha, \beta, \gamma, \delta$, and g are also subject to disturbances, since the two equations in question are really just a part of the larger system consisting of equations (5.1) - (5.5).

In order to handle the complete system, the author devised a metric theory (cf. MP_1 and MP_2) which may be applied to any system of differential equations of the form,

$$dx_i/dt = X^i[x, t] \qquad i = 1, \cdots, n.$$

In this theory we introduce a Riemann metric (in general varying with the time t) for the n dimensional space of the x's. An inequality is then developed for the "distance between two solutions" at time t in terms of the distance at time 0. From this inequality rigorous estimates can be read off to answer at least some questions about the way in which the solution depends on initial conditions. Results are certainly obtained no matter how the metric is chosen but, unless the choice is a fortunate one, the results may turn out to be too trivial to be of practical value. A minimum desideratum would be to choose the metric in such a way that it would be sensitive to situations causing stability or instability with regard to the simplified system consisting of (5.4) and (5.5).

For this reason a preliminary investigation was carried out to choose a favorable metric for the study of (4.1). It may be noted in this connection that the choice is not influenced by the presence of the terms H(t) and L(t) in (4.1). This is because the difference between two solutions of the linear non-homogeneous equations always satisfies the homogeneous equations. A theory of choosing a metric for a linear system is indicated in Section 5 of MP_2. When this theory is applied to (4.1), we are led to the following quadratic form for the Riemann metric of the four real dimensional space of $w = \omega_1 + i\,\omega_2$ and $u = u_1 + iu_2$.

$$ds_1{}^2 = |dw|^2 + (2\bar{\gamma})^{-1}(\bar{\delta} - \bar{\alpha})\,dw\,d\bar{u} + (2\bar{\gamma})^{-1}(\delta - \alpha)\,d\bar{w}\,du +$$

$$(|\delta - \alpha|^2 + |(\delta - \alpha)^2 + 4\beta\gamma|)\,|2\gamma|^{-2}\,|du|^2,$$

where $\bar{w}$, $\bar{u}$, $\bar{\alpha}$, $\bar{\gamma}$, etc., are used to indicate the conjugate imaginaries of w, u, α, γ, etc. In another paper[6] this metric was actually used to obtain inequalities for the system (4.1) and these inequalities were shown to be the best possible ones of their type. Hence we are confident that in setting up a favorable metric for the entire system dw and du should enter as in $ds_1{}^2$ or (what leads to the same result) $ds_1{}^2$ multiplied by an arbitrary positive function.

As already indicated, the rest of the system is not so sensitive to disturbing phenomena and hence there seems to be no particular reason for using anything other than the simplest choice of a Euclidean metric:

$$ds_2^{\,2} = du_0^{\,2} + dg_1^{\,2} + dg_2^{\,2} + d\omega_0^{\,2} .$$

It may be noted in this connection that the non linear system of Section 6 of MP_1 is concerned with the corresponding problem of particle exterior ballistics, in which x and y denote two components of the velocity of the particle. Here the use of a Euclidean metric is extremely satisfactory.

Thus, for the complete system one would expect satisfaction from a metric of the form,

$$ds^2 = |2\gamma|^2 ds_1^{\,2} + J ds_2^{\,2}.$$

Here $ds_1^{\,2}$ is multiplied by $|2\gamma|^2$ in order to avoid fractions and J is a positive constant to be chosen at pleasure. Considerable effort has been spent to determine an efficient value for J. Since these efforts have been unavailing, one may suppose that different values may be desired for different applications.

7. <u>The fundamental quadratic form.</u> We turn now to the fundamental result of MP_1 which implies that, if $\mathcal{D}(t)$ denotes the distance between solutions at time t of the system,

$$dx_i/dt = X^i(x_1, \cdots, x_n), \quad i = 1, \cdots, n, \tag{7.1}$$

where "distance" is taken in the Riemannian sense with respect to a specified metric,

$$ds^2 = \sum_{i,j} g_{ij} dx_i dx_j, \tag{7.2}$$

then

$$\mathcal{D}(t) \leq \mathcal{D}(t_0) e^{\bar{\beta}|t - t_0|} \tag{7.3}$$

where $\bar{\beta}$ is an upper bound for a certain quadratic form

$$Q[\lambda] = \sum_{i,j} q_{ij} \lambda_i \lambda_j, \tag{7.4}$$

under the condition $\sum_{i,j} g_{ij} \lambda_i \lambda_j = 1$. Here the coefficients of the "fundamental" quadratic form $Q[\lambda]$ are given by the following formula"

$$q_{ij} = \sum_k (\tfrac{1}{2} X^k \partial g_{ij}/\partial x_k + g_{ik} \partial X^k/\partial x_j). \tag{7.5}$$

In this section we wish to produce a table for the coefficients of this quadratic form for the ballistic equations (5.1) - (5.5) with respect to the metric introduced in the preceding section. First, however, we indicate the change of notation to harmonize MP_1 with the notation of the exterior ballistic problem, and then write out explicitly the differential equations and the coefficients of the chosen metric in the notation of MP_1.

In our problem the n of MP_1 is equal to 8, and we choose $x_1 = \omega_1$, $x_2 = \omega_2$, $x_3 = u_1$, $x_4 = u_2$, $x_5 = u_0$, $x_6 = g_1$, $x_7 = g_2$, $x_8 = \omega_0$. Then the differential equations in the new notation are as follows:

$$\frac{dx_1}{dt} = X^1 \equiv a_{12}x_1x_5 - (a_{21} + h)x_2x_8 + b_{11}x_3x_8 - b_{22}x_4x_5$$

$$\frac{dx_2}{dt} = X^2 \equiv a_{12}x_2x_5 + (a_{21} + h)x_1x_8 + b_{11}x_4x_8 + b_{22}x_3x_5$$

(These two equations were obtained by equating real and pure imaginary part of (5.4)).

$$\frac{dx_3}{dt} = X^3 \equiv c_{11} x_1x_8 - (c_{22} + 1)x_2x_5 + d_{12} x_3x_5 - d_{21}x_4x_8 + x_6$$

$$\frac{dx_4}{dt} = X^4 \equiv c_{11}x_2x_8 + (c_{22} + 1)x_1x_5 + d_{12}x_4x_5 + d_{21}x_3x_8 + x_7$$

(These two equations were obtained by equating real and pure imaginary parts of (5.5)).

$$\frac{dx_5}{dt} = X^5 \equiv x_2x_3 - x_1x_4 - (|G|^2 - x_6^2 - x_7^2)^{\frac{1}{2}} + f_{22}x_5^2$$

$$\frac{dx_6}{dt} = X^6 \equiv x_2 \, (|G|^2 - x_6^2 - x_7^2)^{\frac{1}{2}}$$

$$\frac{dx_7}{dt} = X^7 \equiv - x_1 \, (|G|^2 - x_6^2 - x_7^2)^{\frac{1}{2}}$$

$$\frac{dx_8}{dt} = X^8 \equiv m_{12}x_5x_8$$

We similarly write down the explicit expressions for the $g_{ij} = g_{ji}$

$$g_{11} = 4(c_{11}{}^2\omega_0{}^2 + (c_{22}+1)^2u_0{}^2) = 4(c_{22}+1)^2x_5{}^2 + 4c_{11}{}^2x_8{}^2$$

$$g_{12} = 0$$

$$g_{13} = 2\left[c_{11}(d_{12}-a_{12}) + (c_{22}+1)(d_{21}-a_{21}-h)\right]x_5x_8$$

$$g_{14} = 2(c_{22}+1)(d_{12}-a_{12})x_5{}^2 - 2c_{11}(d_{21}-a_{21}-h)x_8{}^2$$

$$g_{21} = 0,\ g_{22} = g_{11},\ g_{23} = -g_{14},\ g_{24} = g_{13}$$

$$g_{31} = g_{13},\ g_{32} = -g_{14}$$

$$g_{33} = (d_{12}-a_{12})^2x_5{}^2 + (d_{21}-a_{21}-h)^2x_8{}^2 + \left[\mathcal{A}_1{}^2x_5{}^4 + (2\mathcal{A}_1\mathcal{A}_2 + \mathcal{A}_3{}^2)x_5{}^2x_8{}^2 + \mathcal{A}_2{}^2x_8{}^4\right]^{\frac{1}{2}}$$

where

$$\mathcal{A}_1 = (d_{12}-a_{12})^2 - 4b_{22}(c_{22}+1),\quad \mathcal{A}_2 = 4b_{11}c_{11} - (d_{21}-a_{21}-h)^2$$

$$\mathcal{A}_3 = 2(d_{12}-a_{12})(d_{21}-a_{21}-h) + 4b_{11}(c_{22}+1) + 4b_{22}c_{11}$$

$$g_{34} = 0,\ g_{41} = g_{14},\ g_{42} = g_{13},\ g_{43} = 0,\ g_{44} = g_{33}$$

$$g_{55} = g_{66} = g_{77} = g_{88} = J,$$

$g_{ij} = 0$ if at least one of the subscripts is > 4 and if $i \neq j$.

On account of relations, $\partial X^2/\partial x_2 = \partial X^1/\partial x_1$, $\partial X^3/\partial x_2 = -\partial X^4/\partial x_1$, $\partial X^3/\partial x_1 = \partial X^4/\partial x_2$, $\partial X^1/\partial x_2 = -\partial X^2/\partial x_1$, etc., which are readily verified from the formulas for the X's, we find with the help of (7.5) that

$$q_{11} = q_{22},\ q_{12} = -q_{21},\ q_{33} = q_{44},\ q_{13} = q_{24}$$

$$q_{14} = -q_{23},\ q_{31} = q_{42},\ q_{32} = -q_{41},\ q_{34} = -q_{43}.$$

These relations merely imply that the part of the quadratic form $Q[\lambda]$ involving only $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is a Hermitian form in $\lambda_1 + i\lambda_2$ and $\lambda_3 + i\lambda_4$; and this is just what we might have expected from the pedigree of $Q[\lambda]$. It is not necessary to compute either q_{12} or q_{34}, since it is only the symmetrical part of the matrix (q_{ij}) which is essential. Hence, for $i,j = 4$, there are only six quantities to be computed, namely

$$(7.5.1)\quad q_{11} = q_{22} = 4(c_{22}+1)^2 x_5\left[x_2x_3 - x_1x_4 - (|G|^2 - x_6^2 - x_7^2)^{\frac{1}{2}}\right] + 2(c_{22}+1)^2(a_{12}+d_{12}+2f_{22})x_5^3 + 2c_{11}^2(a_{12}+d_{12}+2m_{12})x_5x_8^2.$$

$$(7.5.2)\quad q_{13} = q_{24} = (c_{11}(d_{12}-a_{12}) + (c_{22}+1)(d_{21}-a_{21}-h))x_8(x_2x_3 - x_1x_4 - R_1) + \Big[(f_{22}+2d_{12}+m_{12})(c_{11}(d_{12}-a_{12}) + (c_{22}+1)(d_{21}-a_{21}-h)) + 4b_{11}(c_{22}+1)^2 + 2d_{21}(d_{12}-a_{12})(c_{22}+1)\Big]x_5^2x_8 + \left[4b_{11}c_{11}^2 - 2c_{11}d_{21}(d_{21}-a_{21}-h)\right]x_8^3,$$

where we have set $R_1 = (|G|^2 - x_6^2 - x_7^2)^{\frac{1}{2}}$.

$$(7.5.3)\quad q_{31} = q_{42} = (c_{11}(d_{12}-a_{12}) + (c_{22}+1)(d_{21}-a_{21}-h))(x_2x_3 - x_1x_4 - R_1)x_8 + \Big[(f_{22}+m_{12}+2a_{12})(c_{11}(d_{12}-a_{12}) + (c_{22}+1)(d_{21}-a_{21}-h) - 2(c_{22}+1)(d_{12}-a_{12})(a_{21}+h) + c_{11}(d_{12}-a_{12})^2\Big]x_5^2x_8 + c_{11}(d_{21}-a_{21}-h)(d_{21}+a_{21}+h)x_8^3 + c_{11}R_2x_8,$$

where we have set $R_2 = (\mathcal{A}_1^2x_5^2 + (2\mathcal{A}_1\mathcal{A}_2 - \mathcal{A}_3^2)x_5^2x_8^2 + \mathcal{A}_2^2x_8^2)$.

$$(7.5.4)\quad q_{14} = -q_{23} = 2(c_{22}+1)(d_{12}-a_{12})(x_2x_3 - x_1x_4 - R_1)x_5$$

$$+ 2(c_{22}+1)\left[(d_{12}-a_{12})(f_{22}+d_{12}) - 2b_{22}(c_{22}+1)\right]x_5^3$$

$$-2\left[(d_{21}-a_{21}-h)(c_{11}m_{12} + (c_{22}+1)d_{21} + c_{11}d_{12}) + 2b_{22}c_{11}^2 + c_{11}d_{21}(d_{12}-a_{12})\right]x_5x_8^2$$

$$(7.5.5)\quad q_{41} = -q_{32} = 2(c_{22}+1)(d_{12}-a_{12})(x_2x_3 - x_1x_4 - R_1)x_5$$

$$+ (c_{22}+1)(d_{12}-a_{12})(2f_{22}+a_{12}+d_{12})x_5^3$$

$$+\left[2c_{11}(a_{21}+h)(d_{12}-a_{12}) - (d_{21}-a_{21}-h)(2c_{11}m_{12} + 2a_{12}c_{11} - (a_{21}+d_{21}+h)(c_{22}+1))\right]x_5x_8^2$$

$$+ (c_{22}+1)R_2x_5.$$

$$(7.5.6)\quad q_{33} = q_{44} = \tfrac{1}{2}(x_2x_3 - x_1x_4 - R_1)\left[2(d_{12}-a_{12})^2x_5 + \frac{\partial R_2}{\partial x_5}\right]$$

$$+ \tfrac{1}{2} f_{22}x_5^2\frac{\partial R_2}{\partial x_5} + \tfrac{1}{2} m_{12}x_5x_8\frac{\partial R_2}{\partial x_8} + d_{12}R_2x_5$$

$$+ (d_{12}-a_{12})((d_{12}+f_{22})(d_{12}-a_{12}) - 2b_{22}(c_{22}+1))x_5^3$$

$$+\left[(d_{21}-a_{21}-h)\Big((d_{21}-a_{21}-h)(m_{12}+d_{12}) + 2b_{11}(c_{22}+1) + 2b_{22}c_{11}\Big) + 2b_{11}c_{11}(d_{12}-a_{12})\right]x_5x_8^2.$$

The other q_{ij} are relatively simple to calculate from (7.5) and will be left to the reader. The complexities of formulas (7.5.1) to (7.5.6) make it seem unlikely that any simple general conclusions can be drawn. The method gives primarily a procedure for the numerical study of individual cases.

8. A word of warning. After writing the above material, it gradually came to the author's attention that the elimination of g_0 by means of $g_0^2 + g_1^2 + g_1^2 + g_2^2 = |G|^2$ introduces a difficulty in the study of a trajectory near its vertex, that is, where the axis of the shell is nearly horizontal. This is because the derivative of the function R_1 with respect to x_6 or x_7 is very large when g_0 or R_1 is close to 0. Hence the method, as worked out above, is pertinent largely to antiaircraft fire or other types of trajectories where the shell axis is never horizontal in the

important part of the trajectory. If this is not the case, the theory may be modified in one of two ways. Either eliminate g_2 instead of g_0 (this has the disadvantage of making our equations more unsymmetrical) or else refrain from eliminating any of the g's and use a system of order nine instead of one of order eight. In either event, the modifications necessary for this purpose are only slight, but it is not possible to work out the details here.

FOOTNOTES

1. Cf. D. C. Lewis, "Metric properties of differential equations," MP_1 American Journal of Mathematics, vol. 71 (1949), pp. 294-312, and "Differential equations referred to a variable metric," MP_2, ibid., vol. 73 (1951), pp. 48-58.

2. The use of the so-called "spinless frame" was suggested by A. W. Wundheiler.

3. Cf. Nielsen and Synge. "On the motion of a spinning shell," Quarterly of Applied Mathematics, vol. 4 (1946), pp.201-226.

4. As is well known (3.2) can also be obtained from (3.1) even when the latter is known to hold for only a single value of $\alpha \not\equiv 0 \bmod \pi$. Hence symmetry under rotations through $\alpha = \frac{2\pi}{n}$, $n > 2$, also lead to the form (3.2). Complete axial symmetry is not necessary.

5. Cf. Nielsen and Synge, loc. cit.

6. Cf. D. C. Lewis, "Inequalities for complex linear differential systems of the second order." Proceedings of the National Academy of Sciences, vol. 38 (1952), pp. 63-66.

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