OFFICE OF NAVAL RESEARCH

Contract N7onr-35801

T. O. I.

NR-041-032

Technical Report No. 106

PLASTIC STRESS-STRAIN RELATIONS BASED ON INFINITELY MANY PLANE LOADING SURFACES

by

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BROWN UNIVERSITY

PROVIDENCE, R. I.

March, 1954

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ABSTRACT

This paper is concerned with the development of a theory of plastic stress-strain relations for work hardening materials based on infinitely many plane loading surfaces. The stress-strain relations belonging to this class are closely related to those the the linear incremental type but have the property of being integrable in a restricted sense. They are also non-linear in that a corner appears in the yield surface at the point of loading. A stress-strain relation of this type for isotropic materials is presented. The problem of including a description of Bauschinger and allied effects within the theory is considered.

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* The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract N7onr-35801 with Brown University.

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Introduction

Some recent developments [1]*, [2] have made it feasible to investigate the possibilities of a certain class of non-linear incremental stress-strain relations for work hardening materials. Linear incremental stress-strain relations have been extensively investigated and results of a quite general nature have been achieved [3], [4]. Studies of the behavior of metals from a phenomenological point of view have revealed a few conditions which greatly restrict the possible forms of stress-strain relations, linear or not. In particular a work hardening condition postulated by Drucker [2] has for reaching consequences. Under this condition he has shown that the yield surface must be convex, that at any smooth point of the yield surface the strain increment vector must be normal to the surface for loading from that point, and that there are certain restrictions on the directions of the strain increment vector for loading from a corner. From these last restrictions it follows that the stress-strain relations must be non-linear for loading from a corner.

Before these latter results were obtained a stress-strain relation based on physical consideration of a crystal aggregrate had been proposed by Batdorf and Budiansky [5]. It was later shown to satisfy Drucker's work hardening condition and to possess a uniqueness and a variational theorem [6]. This

* Numbers in square brackets refer to Bibliography at the end of the paper.

theory, called the slip theory of plasticity, is non-linear in that a corner is always formed in the yield surface at the loading point. Since the stress-strain relations of slip theory were formulated without mention of a yield surface and since practically all previous theories had been linear (and hence properly applicable only in case of a smooth yield surface) it was thought that slip theory stood in a class by itself.

The work done to obtain theorems regarding the normality of the strain increment vector to the yield surface was partly motivated by a desire to use the corresponding yield function (or loading function) as a plastic potential in the stress-strain relations. Recently Koiter [1] showed that the concept of plastic potential may be retained even if the yield surface is singular (with corners, ridges, etc.) by the simple device of introducing more than one loading function. The yield surface is the boundary of those points in stress space which represent elastic behavior with respect to all loading surfaces and of course the yield surface is singular where two or more loading surfaces intersect. In such a theory of plasticity the total plastic strain increment is the sum of the contributions from each loading function. The stress-strain relation is necessarily non-linear at corners in the yield surface but may possibly be non-linear only at the corners. Certain uniqueness and variational theorems were extended by Koiter to cover this class of non-linear theories.* He also showed that slip theory

* Which also satisfy the work hardening condition and the so-called conditions of continuity and consistency [3].

belongs to this class. The loading surfaces of slip theory are an infinite set of planes.

A qualitative discussion of theories of plasticity based on many plane loading surfaces was given in [7]. It was shown that in certain cases the resulting stress-strain relations are integrable in a restricted sense and thus partially resemble the stress-strain relations of deformation theories of plasticity. The present paper is concerned with further development of the theory. A stress-strain relation for initially isotropic materials is constructed and certain interesting special cases are discussed. It is shown how an account of Bauschinger and allied effects may be included within the theory by postulating a relation between the displacements of the infinitely many loading planes which envelope the yield surface. A function depending on loading history alone is introduced which determines both the yield surface and the plastic strain.

Plane Loading Surfaces in General.

We begin by noting that at least some planes in stress space have a direct physical interpretation. The shear stress in the direction λ_i on an element of area normal to μ_i is given by

$$\tau = \sigma_{ij} \lambda_i \mu_j \tag{1}$$

where λ_i and μ_i are unit vectors. The geometrical representation in stress space of the equation τ = constant is of course a plane.

Suppose that the critical shear stress on a certain slip system in a single crystal is k. In this case $\tau = k$ might be regarded as the initial yield condition for the slip system, the corresponding plane in stress space as the initial yield surface, and the function τ of the stresses (1) as the loading function. One may easily verify that the shear strain due to slipping is represented by a vector normal to this plane (in nine dimensional stress space). If a plane loading surface is introduced for all possible orientations of slip systems in a random aggregate of crystals and certain other assumptions are made than the result is the so-called slip theory of plasticity [5]. In this paper the theory of stress-strain relations for work hardening materials based on an infinite number of plane loading surfaces is considered as a subject in itself. No attempt will be made to develop a rational theory based on the physics of crystal aggregates but some suggestions from that source will be of use.

Suppose that an initial yield surface for a strain hardening material has been given. Such a surface must be convex hence it can be regarded as an envelope of planes which nowhere intersect the surface. For the present assume that each plane behaves as though the others were not there. As loading into the plastic range proceeds each plane is moved outward parallel to itself by the loading point more or less or not at all depending on the loading path. Two examples of the resulting yield boundary for biaxial loading are shown in Figs. 1(a) and 1(b). Each plane remains as far out as it has ever been pushed by the loading point during the course of loading and ordinarily there is a corner in the yield surface at the final load point.* At each stage of loading the yield surface is convex, as it must be for a work hardening material according to a theorem of Drucker [2].

The plastic strain is assumed to be the sum of contributions from each loading plane. The contribution from a single plane is assumed to be given by a linear stress-strain relation of the Prager-Drucker type in which the loading function f is linear and homogeneous in the stresses and where $f = c > 0$ is the equation of the loading plane. The contribution of a group of planes whose normals lie within a small solid angle is given by:

$$\begin{aligned} \delta\, \dot{\varepsilon}_{ij}^{p} &= G(f,\omega)\, n_{ij}\, \dot{f}\, \delta\omega \qquad && \dot{f} > 0 \\ &= 0 && \dot{f} \leq 0 \end{aligned} \tag{2}$$

where $n_{ij} = \frac{\partial f}{\partial \sigma_{ij}}$ is a constant normal vector to one of the loading planes, $\delta\omega$ is an element of solid angle, and $G > 0$ if f exceeds a certain value which is the yield point for this plane but vanishes otherwise. The possible dependence of G upon the orientation of the loading plane is indicated by writing it as a function of ω. Equation (2) may be integrated with respect to time to give:

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* See [8] for recent experimental evidence of corners.

$$\delta \varepsilon_{ij}^{p} = H(f,\omega)\, n_{ij}\, \delta\omega \tag{3}$$

(where $\frac{\partial H}{\partial f} = G$) as may easily be verified by differentiating (3). The total plastic strain is obtained by integrating over all orientations of loading planes:

$$\varepsilon_{ij}^{p} = \int \delta \varepsilon_{ij}^{p} \tag{4}$$

The value of f appearing in (3) is the largest value attained during loading and is a measure of the distance the loading plane has been pushed out. As a result of the integrability of (2) the strain contributed by each loading plane depends on the loading path only in so far as it depends on the distance the loading plane has been pushed out during the course of loading. As a consequence of this integrability the following statement may be made concerning the stress-strain relation (4): "The total plastic strain is the same for any two loading paths which result in the same yield surface." In this sense the stress-strain relation is path independent.* This of course is not the same as saying that the total plastic strain is the same for any two loading paths which reach the same final load point.

Any loading path resulting in a yield surface which could also be produced by a radial loading path will be called a "nearly radial loading path." In Fig. 2 the nearly radial loading path OPQ results in the same yield surface as the radial path OP'Q. Note that a path which is not nearly radial (OP"P')

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* In the case of slip theory this theorem was known to its authors.

may possibly be made so by further loading (OP"P'Q). In Fig. 3 the loading point has reached P by a nearly radial loading path. If the loading path proceeds into the shaded region then all planes previously loaded continue to load. This region will be called the region of total loading. Any loading path which always proceeds into the region of total loading is nearly radial.

So far it has been assumed that each plane acts independently of the others and that no plane moves unless the loading point moves it. Under such an assumption the yield point in compression is unaltered by loading into the plastic range in tension. This however is not in accord with experiment. Often hardening in tension produces a softening in compression known as the Bauschinger effect. The yield points in tension and compression at right angles to the original direction of loading may also be affected (cross effect). In theory at least the whole yield surface can expand uniformly. In general of course the whole yield surface is affected in some way or other by any kind of loading and not just part of the yield surface as was indicated in Fig. 1. In order to produce these effects it is necessary to assume some sort of interdependence among the loading planes.

The number associated with a plane, which completely determines its position, is the distance of the plane from the origin (exactly how distance is to be defined in stress space will be discussed later). It seems best to abandon the use of a loading function as such and to concentrate attention upon the changes in distance* of the planes from the origin as loading proceeds. The desired interdependence between the loading planes may be achieved by postulating a relation between the motions of the various planes. It will be assumed that the loading of a given plane induces a motion in any other plane according to some rule which involves the orientations of the two planes and the measure of loading of the given plane. Only those planes in contact with the loading point may be said to be loading, thus the motion of the loading point completely determines the total motion of the planes being loaded. However, the planes being loaded induce a motion in each other which when subtracted from their total motion leaves a remainder to be accounted for. This remainder which is called the direct motion will be taken to be the measure of loading. At this time the reader is asked to accept the following statement on faith. If a suitable relation between the direct and induced motions is given, and the loading path is given, then the requirement that the direct motion shall always be in the loading (outward) sense results in a determinate system. A partial justification of this statement is made later in the paper. The situation is a little clearer where the induced motion is very small for then the yield surface is almost the same as though the planes moved independently.

There is at least an intuitive connection between these notions and the behavior of crystal aggregates. During plastic

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* Distance in stress space is defined later.

straining the inactive slip systems are hardened as well as the active ones, a phenomenon known as latent strain hardening. This suggests the induced motion of loading planes. That it is possible to obtain reasonable results based on the above assumptions has been shown by actually constructing a few examples.

In these examples the type of behavior exhibited by some known stress-strain relations possessing a Bauschinger effect [9], [10], [11] has been duplicated. The yield boundary after biaxial loading according to a typical theory of this kind is shown in Fig. 4.* The initial yield boundary might be the J_2 ellipse. After loading in tension to P this ellipse is enlarged and displaced to the right. In an example constructed using plane loading surfaces acting interdependently the behavior illustrated in Fig. 5 was obtained. The greater part of the yield boundary is modified in the same way as before but now a corner appears at the loading point P. By varying certain parameters involved the corner at P may be made as blunt as we please. The behavior shown in Fig. 1 and in Fig. 4 may be obtained as limiting cases. The yield boundary according to simple J_2 flow theory is also obtainable as a limiting case.

In the stress-strain relations for this theory the direct motion of a loading plane is taken as the measure of its contribution to the plastic strain rather than the total motion so that for a small group of planes:

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* Drucker's [2] definition of a work hardening material does not require the origin to be inside the yield surface.

$$\delta\, \varepsilon_{ij}^{p} = H(r_1,\omega)\, n_{ij}\, \delta\omega \qquad (5)$$

where r_1 is the direct motion. The integrated form of (5) naturally raises the question of path independence. About the best that can be said at present is that if the yield surface determines the direct motion then the total plastic strain is the same for any two loading paths which result in the same yield surface. This amounts almost to a tautology. The difficulty is that although the direct motion determines the yield surface the converse is not true unconditionally. Certainly any unqualified assertion concerning path independence must be false in some of the limiting cases. However there are cases in which path independence is possible.

Figure 6 illustrates a case based on the examples constructed in a later section in which two different loading paths OPQ and OP'Q result in the same yield surface and total plastic strain. There is a region of total loading beyond Q into which a nearly radial loading path may proceed; it is shown bounded by the dashed lines. In the limiting cases in which the corner becomes blunted the region of total loading narrows down to a line so that nearly radial loading becomes strictly radial loading.

The mathematical formulation of the problem of finding the yield surface according to the present theory involves an integral equation. The conditions under which the problem has a unique solution have not yet been found. There is no doubt about uniqueness in the example solutions given but only nearly radial loading paths have been considered. Even if suitable conditions are imposed to insure the unique determination of the direct motion it remains to be shown that the corresponding stress-strain relations lead to unique solutions of boundary value problems. In order to extend the uniqueness theorem to cover the case of interdependent motion* it may be that some additional restrictions must be placed on the rule governing the relation between the direct motion and the induced motion.

The problem of determining the yield surface for loading paths which are not nearly radial is considerably more complicated than in the case in which the loading planes move independently. The yield surface resulting from such a loading path might look something like Fig. 7 in which OPQ is a broken line with a sufficiently large turning angle at P. A corner previously formed at P has been rounded off and a new corner formed at Q.

The present theory affords a means of taking Bauschinger and allied effects into account without explicitly introducing strain into the loading functions as has been the usual practice to date. However the attendent complications in determining the yield surface have not been avoided. Although it is still convenient to speak of loading planes there doesn't seem to be anything left which could strictly be called a loading function. The direct motion is sort of an intermediate variable determined by the loading history and which in turn determines

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* Koiter's theorem [1] is applicable in the case of independent motion.

both the yield surface and the plastic strain.

Stress-Strain Relation for Isotropic Materials.

In this section the loading planes are assumed to move independently. A general stress-strain relation for isotropic materials is constructed. The stress-strain relations for slip theory and for a theory which agrees with J_2 deformation theory for nearly radial loading are written down as special cases.

In dealing with plane loading surfaces a certain amount of mathematical machinery of a geometrical nature is almost indispensable for concise expression of the ideas involved. Once a suitable stress space has been constructed, and some appropriate coordinate system has been defined in this space, writing down a stress-strain relation for isotropic materials based on infinitely many plane loading surfaces is almost a trivial matter. Since plastic strain is assumed to depend only on the stress deviator, which has five independent components, a stress space of five dimensions is indicated. For the five cartesian coordinate members it is possible to choose linear combinations of the stress deviator such that a "stress vector" in stress space transforms like a vector when the corresponding stress tensor is transformed by a rotation of the physical coordinates. The length of the stress vector, being an invariant, is not changed by the transformation. This inevitably leads to a distance function in stress space proportional to* $(J_2)^{\frac{1}{2}}$. In such a stress space any quantity such as distance, angle, area,

* $J_2 = \frac{1}{2} s_{ij} s_{ij}$.

etc. defined in terms of the metric may be used consistently because it is an invariant with respect to changes of coordinates in physical space. A stress space of this kind is constructed in appendix A. Several systems of coordinates for this stress space are defined in the appendix which also contains other details of a geometrical nature. Only a brief description of these results will be given in this section as the need arises.

The contribution to the total plastic strain of a group of planes which were initially tangent to the initial yield surface over an element of area da is assumed to be given by:

$$\delta e_\alpha = H(f, n_\alpha)\, n_\alpha\, da \quad (\alpha = 1,2,..,5) \tag{6}$$

where e_α is the plastic strain vector, and n_α is the unit normal vector to the element of area directed in the loading sense. The stress vector is denoted by x_α and f is given by $f = n_\alpha\ x_\alpha$, or rather the maximum value of this quantity attained during the course of loading. $H > 0$ if $f > n_\alpha\ R_\alpha$ where R_α is the vector from the origin to a point within the element of area da on the initial yield surface, otherwise H vanishes. The contributions of all planes are summed to get the total plastic strain:

$$e_\alpha = \int_S H\, n_\alpha\, da \tag{7}$$

S is the initial yield surface which is the envelope of the initial positions of all the loading planes. The stress-strain relation (7) is in general anisotropic; further specialization is necessary to arrive at a relation applicable to initially isotropic materials.

In appendix A a special spherical coordinate system is defined in which

$$J_2 = \rho^2 = x_\alpha x_\alpha$$

and (8)

$$J_3 = \frac{2}{3\sqrt{3}} \rho^3 \cos 3\Theta_4$$

Three other coordinate angles Θ_1, Θ_2, Θ_3 are introduced and the transformation of coordinates is given by equations of the form:

$$x_\alpha = \rho\ \xi_\alpha\ (\Theta_p) \qquad (p = 1,2,3,4) \tag{9}$$

where $\xi_\alpha \xi_\alpha = 1$. The equation of the initial yield surface of an isotropic material in these coordinates is given by:

$$\rho = P(\Theta_4) \tag{10}$$

Isotropy requires that all planes be treated equally which are initially tangent to the initial yield surface along its intersection with a coordinate surface Θ_4 = constant. The special form of the stress-strain relation (7) for initially isotropic materials is thus:

$$e_\alpha = \int_s H(f,\Theta_4)\ n_\alpha\ da \tag{11}$$

or more explicitly:

$$e_\alpha = \int_s H(f,\Theta_4)(\xi_\alpha - \frac{P'}{P} \frac{\partial \xi_\alpha}{\partial \Theta_4}) P^4 \sin \Theta_2 \sin 3\Theta_4\ d\Theta_1\ d\Theta_2\ d\Theta_3\ d\Theta_4\ . \tag{12}$$

where $H = 0$, $f < P\, n_\alpha\, \xi_\alpha$. Sometimes some other coordinate system is more convenient than the one used in (11); in a general system of coordinates this relation may be written:

$$e_\alpha = \int_s H(f, J_3^*)\, n_\alpha\, da \tag{13}$$

where J_3^* is the value of J_3 at the point of tangency to the yield surface of the loading plane in its initial position.

The simplest special case of the stress-strain relation constructed above is that in which the initial yield surface is $\rho = k$ and all planes are treated equally, in this case:

$$e_\alpha = \int_s H(f)\, \xi_\alpha\, da \qquad (H = 0,\ f < k) \tag{14}$$

The initial yield surface is of course the Mises yield surface which is a sphere in the present stress space. For radial loading (or nearly radial loading for that matter) the strain vector must have the same direction as the stress vector because of the perfect symmetry of the configuration of loading planes. Again because of symmetry the length of the strain vector can only depend on the length of the stress vector and not upon its direction. The stress-strain relation for nearly radial loading must be:

$$e_\alpha = F(x_\beta x_\beta)\, x_\alpha \tag{15}$$

In the more usual notation this is

$$\varepsilon_{ij}^{p} = F(J_2)\, s_{ij} \tag{16}$$

which is immediately recognized as the stress-strain relation of J_2 deformation theory. The form of the stress-strain relation for a broken line loading path with a large turning angle is given in appendix B.

Another interesting special case is slip theory. The loading planes of slip theory in their initial positions are given by:

$$f = \sigma_{ij} \lambda_i \mu_j = \tfrac{1}{2} s_{ij} (\lambda_i \mu_j + \lambda_j \mu_i) = k \tag{17}$$

where λ_i and μ_i are all possible pairs of orthogonal unit vectors in physical space [1] . It is easy to verify that the vector:

$$s^{o}_{ij} = k(\lambda_i \mu_j + \lambda_j \mu_i) \tag{18}$$

drawn from the origin, is normal to the corresponding plane (in 9 dimensional stress space) and moreover terminates in the plane. Also $\frac{1}{2} s^{o}_{ij} s^{o}_{ij} = k^2$ and $\frac{1}{3} s^{o}_{ij} s^{o}_{jk} s^{o}_{ki} = 0$. Therefore the planes of slip theory are all those planes tangent to the surface $J_2 = k^2$ where $J_3 = 0$ and no others. The same holds in x stress space. All these planes are to be treated equally because slip theory applies to an initially isotropic material. Here is a case in which not all planes available in stress space are put to use. Even though all the planes used are initially tangent to the sphere $\rho = k$ the initial yield surface, as is well known, is the Tresca yield surface. The stress-strain relation is obtained from (12) by omitting the integration with respect to Θ_4.

$$e_\alpha = \int_{S'} H(f)\, \xi_\alpha \sin \theta_2 \, d\theta_1 \, d\theta_2 \, d\theta_3 \quad (H = 0, f < k) \qquad (19)$$

where S' is the intersection of $J_2 = k^2$ with $J_3 = 0$. The parameters $\theta_1, \theta_2, \theta_3$ involved in the original formulation of slip theory [5] were somewhat different.

In both of the preceding cases the function H(f) may be obtained from a stress-strain curve by solving (in closed form) a simple type of integral equation. In the general case more experimental information than a stress-strain curve would be needed to determine $H(f, \theta_4)$, and solving the corresponding integral equation would not be a simple matter. Of course a form for H could be assumed involving a number of arbitrary parameters and then one could attempt to match the data by adjusting the parameters, but at present it doesn't seem to be worth while to pursue the matter any further.

If the loading path is such that the principal axes of stress do not rotate then the loading path is confined to the x_1, x_2 plane provided the physical coordinates are referred to the principal axes. Again if the only stresses imposed are say σ_{11} and σ_{12} then the loading path is confined to the x_1, x_3 plane. In these cases the stress-strain relations based on loading planes in stress space can be reduced to a form having the appearance of a stress-strain relation based on "loading lines" in a "stress plane". The reduction can always be made, at least in theory. For simplicity the argument will be presented only for the case in which the initial yield surface is

$\rho = k$ and for the sake of being definite suppose the loading path is confined to the x_1, x_2 plane.

Consider a plane which has the trace t in the x_1, x_2 plane before loading begins (see Fig. 8). Let OP be the perpendicular from O to t and let θ_1 be the angle between OP and the x_1 axis. Let OQ be the perpendicular from O to the plane in question and let θ_2 be the angle POQ. Suppose that during the course of loading this plane is pushed out so that its trace is now t'. The distance of the plane from the origin which was $\overline{OQ} = k$ is now $\overline{OQ'} = \overline{OP'} \cos\theta_2 = f$. Consider now a spherical coordinate system in which the x_1 and x_2 coordinates of Q are given by:

$$\begin{aligned} x_1 &= k \cos\theta_2 \cos\theta_1 \\ x_2 &= k \cos\theta_2 \sin\theta_1 \end{aligned} \qquad (20)$$

and x_3, x_4, x_5 are given in terms of θ_1, θ_2 and two more parameters θ_3, θ_4 (see for example appendix A Eq. A-28). Now since there is a two parameter set of points Q on the sphere, all with the same θ_1 and θ_2 coordinates, there is a two parameter family of planes tangent to the sphere with the same trace t in the x_1, x_2 plane as the plane considered above. The contribution of these planes to the plastic strain is given by:

$$\delta^2 e_\beta = \int_0^{2\pi}\int_0^{\pi} H(f, J_3^*)\ \overline{\xi}_\beta \sin^2 2\theta_2 \sin\theta_3\, d\theta_3\, d\theta_4\, d\theta_1\, d\theta_2$$

$$= \overline{H}(f, \theta_1, \theta_2)\ \overline{\xi}_\beta \sin^2 2\theta_2\, d\theta_1\, d\theta_2 \quad (\beta = 1,2) \qquad (21)$$

where $\bar{\xi}_1 = \cos\theta_1$, $\bar{\xi}_2 = \sin\theta_1$ and $\bar{H} = 0$, $f < k$. For values of θ_2 between 0 and $\pi/2$ there is a continuous distribution of parallel traces each representing a two parameter family of planes.* To obtain the strain due to moving out all those planes whose traces fall between the one tangent to the circle and the one at t' requires the integration of $\delta^2 e_\beta$ with respect to θ_2 between the limits 0 and $\cos^{-1}(k/\overline{OP'})$. Let $\overline{OP'} = \bar{f}$, then

$$\delta e_\beta = \int_0^{\cos^{-1}(k/\bar{f})} = \int_0^{\frac{\pi}{2}} \bar{H}(\bar{f}\cos\theta_2, \theta_1, \theta_2)\bar{\xi}_\beta \sin^2 2\theta_2 \, d\theta_2 \, d\theta_1$$

$$= \bar{\bar{H}}(\bar{f}, \theta_1)\,\bar{\xi}_\beta \, d\theta_1 \; ; \quad \bar{\bar{H}} = 0, \; f < k \; ; \; (\beta = 1,2) \qquad (22)$$

δe_β may be interpreted as the strain due to moving out a "loading line" initially tangent to the yield boundary, moreover δe_β is normal to this loading line. So long as the loading is in the x_1, x_2 plane $\delta e_3 = \delta e_4 = \delta e_5 = 0$ for an isotropic material, otherwise the remaining strains may be expressed as single integrals if so desired. Note that up to this point the exact shape of the (plane) loading path has not entered into the calculations. To complete the analogy the plastic strain e_β is given as the sum of the contributions from all the loading lines.

$$e_\beta = \int_0^{2\pi} \bar{\bar{H}}(\bar{f}, \theta_1)\, d\theta_1 \; ; \; \bar{\bar{H}} = 0, \; \bar{f} < k \; ; \; (\beta = 1,2) \qquad (23)$$

In appendix B the foregoing reduction is carred out for the case in which all planes are treated equally. A similar reduction to two dimensional loading planes in a three dimensional stress space could be made.

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* The trace tangent to the circle represents one plane only.

Induced Motion.

In the first section it was pointed out that a realistic theory of stress-strain relations based on plane loading surfaces should provide for some sort of interdependent motion of planes to properly account for the changes in shape of the yield surface as loading proceeds. A few assumptions regarding the nature of this interdependence were made. Loading of any given plane was assumed to cause an induced motion in all other planes. It was proposed to separate the motion of a plane into two parts, namely direct and induced, where the direct motion is associated with the loading of a given plane and the induced motion represents the effects on the given plane of loading on all other planes. In this section enough analytical detail is supplied to apply the theory to a few simple cases.

The contribution to the induced motion of a given plane due to the direct motion of an arbitrary small group of planes is assumed to be:

$$\begin{aligned} \delta r_2(\theta) &= F[r_1(\theta'), \theta, \theta']\, da \qquad (r_1 > 0) \\ &= 0 \qquad\qquad\qquad\qquad\quad (r_1 = 0) \end{aligned} \tag{24}$$

where θ and θ' denote the orientations of the given and arbitrary planes respectively. The total induced motion of the given plane is obtained by summing the contributions from all planes:

$$r_2 = \int_s F(r_1, \theta, \theta')\, da \tag{25}$$

The total distance of any loading plane from the origin is given by:

$$r = r_o + r_1 + r_2 \tag{26}$$

where r_o is the initial distance from the origin. In the first section certain other assumptions were made concerning the direct motion. These were that only those planes moving in contact with the loading point could receive any direct motion and that direct motion is a non-decreasing function of time as loading proceeds. In symbols these are:

$$r - n_\alpha x_\alpha > 0 \Rightarrow \dot{r}_1 = 0 \tag{27}$$

$$\dot{r}_1 \geq 0 \tag{28}$$

It is understood that:

$$r - n_\alpha x_\alpha \geq 0 \tag{29}$$

Note that:

$$\dot{r} - n_\alpha \dot{x}_\alpha = 0 \not\Rightarrow \dot{r}_1 > 0 \tag{30}$$

In addition to these assumptions it is expected that certain restrictions must be placed upon F in order to obtain a consistent theory.

Two examples are constructed in the following to check the general reasonableness of the assumptions so far and to gain some insight into the nature of the problem of determining the yield surface according to the present scheme. To keep the calculations as simple as possible the initial yield surface is taken to be $\rho = 1$ and the form of (25) is taken to be:

$$r_2 = \int_s r_1(\lambda + \mu \cos \psi) \, da \tag{31}$$

where λ and μ are constants and ψ is the angle between the normals to the planes whose orientations were denoted by Θ and Θ'.

In the first example take $\mu = 0$, $\lambda > 0$ and let the loading path be coincident with the positive x_1 axis. From (31) r_2 is a non-decreasing function of time alone. Thus the yield surface expands uniformly except for the portion of the surface affected by direct motion. As soon as the loading point reaches $x_1 = 1$ some planes begin to load. Assume for the moment that once a plane begins to load it does not subsequently unload. Then at any instant all planes which have been loaded pass through the loading point and all planes which have not been loaded are tangent to the expanding spherical part of the yield surface; the planes just beginning to load do both. The picture is as in Fig. 9. A corner in the yield surface appears at the loading point P. The radius of the spherical part of the yield surface as a function of the load is calculated in appendix C. As $\lambda \to \infty$ the corner at P becomes blunter and vanishes in the limit. In the appendix it is shown that for λ finite the loading path may turn through a certain angle without unloading any planes which have already been loaded. Suppose the loading point follows such a bent path, then at any instant the situation is exactly the same as before except that P lies a little ways off the x_1 axis. The same configuration of loading planes would have been produced had the loading point reached P by a radial path so the direct motion of any plane is the same in either case. As $\lambda \to \infty$ the allowable turning angle goes to zero so there is no path independence in that case. If $\lambda = 0$ then there is no induced motion and we have the case of the last section. If λ is negative there is the possibility of two configurations of loading planes for a given value of the load. Proper restrictions on F should rule out negative values of λ, at least when $\mu = 0$.

For the second example take $\lambda = 0$ and $\mu > 0$; (31) becomes

$$r_2 = \mu \int_s r_1 \cos \psi \, da = \mu \int_s r_1 \, n_\alpha \, n_\alpha' \, da$$

$$= \mu \, n_\alpha \int_s r_1 \, n_\alpha' \, da = n_\alpha \, A_\alpha \qquad (32)$$

where A_α are constants. There is a simple geometrical interpretation of the induced motion in this case. Suppose the yield surface has attained some irregular shape through loading and then additional loading takes place. What is the change in shape of the yield surface due to the additional induced motion alone? Consider a plane tangent to the yield surface at some arbitrary point P. Suppose P is displaced to P' by the displacement vector A_α carrying the plane along with it (see Fig. 10).

The distance the plane has been displaced parallel to itself is $n_\alpha A_\alpha$. Thus the induced motion given by (32) has the effect of displacing the whole yield surface like a rigid body. For μ positive the displacement is in the same general direction as the additional loading. Of course the direct motion further distorts the yield surface in the vicinity of the loading point. The behavior for radial loading is similar to that of the first example, that is for μ finite a corner appears at the loading point and again nearly radial loading paths exist. In this case also the corner disappears for $\mu = \infty$ and nearly radial becomes strictly radial. The calculations are given in appendix C. For sufficiently large negative values of μ there is the possibility of two configurations of loading planes for a given value of the load. The reader may notice other inconsistencies for negative values of λ and μ; at any rate these examples show that there is a definite need for restrictionson F.

A combination of these two examples gives the type of behavior referred to in the first section. Here again the stress-strain relation would agree with J_2 deformation theory for nearly radial loading paths.

Observations and Conclusions.

The plane loading surface has been used as the fundamental building block, so to speak, in the construction of stress-strain relations of a quite general type. Considerable flexibility has been gained by assuming the displacements of the various planes to be interrelated. In many special cases of the present theory the plastic strains are path independent to a certain extent and a special class of loading paths exist, called nearly radial, for which the stress-strain relations reduce to those of a deformation theory. Nearly radial loading paths were shown to exist in all cases not involving induced motion and even in some cases where induced motion of the loading planes is allowed. It is quite probable that nearly radial loading paths exist whenever the function F is bounded. Further theoretical investigations should be made to settle the question. Certainly the validity of the theory should be tested by comparison with experiment.

It was remarked in an earlier section that the yield surfaces of some known theories could be duplicated by limiting cases of the present theory. It is extremely interesting to look into the corresponding question concerning the two stress-strain relations. Only a cursory examination is necessary to show that the known stress-strain relations cannot be obtained as limiting cases of the present theory. Consider the case of the first example discussed in the text and let λ be very large; then the yield surface is very nearly that according to simple J_2 flow theory. Suppose loading in tension has proceeded up to a certain point and then the load is removed. Now compare the stress-strain curves for loading in compression according to J_2 flow theory and according to the present theory. According to J_2 flow theory there will be a sharp break in the stress-strain curve at the yield point, but according to the present theory the stress-strain curve will be smooth because the planes tangent to the sphere on the compression side have not been previously loaded. At first sight this may seem distressing, one might have hoped that the present theory would be general enough to include such a classic example as J_2 flow theory as a limiting case. However, the experimental facts are in favor of the present theory, qualitatively at least. Stress-strain curves in compression following stressing in tension are always found to be quite smooth [12] . Even those linear incremental stress-strain relations which exhibit a Bauschinger effect predict a sharply defined yield point for such an experiment as that described above.

More complicated experiments have been performed in which the material is first stressed in tension and then loaded in some way other than mere compression. The experiments of Taylor and Quinney [13] and of Klingler and Sachs [14] were of this type. In all cases the reported stress-strain curves for the second loading are quite smooth as would be predicted by a stress-strain relation of the type considered in this paper. On the other hand it is a well known fact that the stress-strain curve for a second loading in tension does have a rather sharply defined yield point. This singular behavior at a point on the yield surface where loading has recently occured may be regarded as evidence for corners in the yield surface. The fact that any intermediate loading (before the second tension test) is likely to destroy the sharpness of the yield point is also evidence in support of a theory of the present type. For some reason (possibly creep) the stress-strain curve for reloading in tension does not always have a <u>perfectly</u> sharp break at the yield point; the same reason may explain why corners have not always been found by those experimenters who have looked for them [8], [15].

Certainly the present theory is far from being well developed and much remains to be done before it can even claim to be acceptable. Even so fundamental a condition as the work hardening condition is not automatically satisfied unless restrictions are placed on the rule governing the induced motion. These restrictions, and possibly others necessary to insure uniqueness, are yet to be discovered.

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Appendix A

Five-dimensional stress space.

If plastic strains are assumed to be incompressible then the stress-strain relations and the yield surface depend only on the stress deviator and not upon $J_1 = \sigma_{ii}$. This means that in the usual nine-dimensional stress space the yield surface is a cylinder normal to the plane $\sigma_{ii} = 0$. Rather than represent stress in this nine-dimensional space in which each component of stress acts as a coordinate it is more convenient for present purposes to choose some other system of coordinates which better fits the yield surface. Since the stress deviator has only five independent components it is possible to represent it geometrically in a suitably chosen space of five dimensions. Methods similar to the one given below for constructing a stress space exist in the literature [16], [17] but they are usually not given in much detail so the one used in this paper will be discussed in this appendix.

The stress space chosen here is a subspace of the usual nine-dimensional stress space. The coordinates in this subspace are chosen in the following way. The x_6 axis (which will later be ignored) is taken normal to the plane $\sigma_{ii} = 0$. The x_1 axis is the projection of the σ_{11} axis on the plane $\sigma_{ii} = 0$ and the x_2 axis is taken normal to the x_1 and x_6 axes so that x_1, x_2, x_6 forms a right-handed system of coordinates. The scale of measurement on the x_1, x_2, x_6 axes is distorted by a factor of $\sqrt{2}$ for a reason which will be given later. The equations of transformation between the two systems of coordinate are:

$$x_1 = \frac{1}{2\sqrt{3}} (2\sigma_{11} - \sigma_{22} - \sigma_{33})$$

$$x_2 = \frac{1}{2} (\sigma_{22} - \sigma_{33}) \qquad \text{(A-1)}$$

$$x_6 = \frac{1}{\sqrt{6}} (\sigma_{11} + \sigma_{22} + \sigma_{33})$$

Three more axes x_3, x_4, and x_5 are added to accommodate shears:

$$x_3 = \tau_{12}$$

$$x_4 = \tau_{23} \qquad \text{(A-2)}$$

$$x_5 = \tau_{13}$$

so that x_1, x_2 ..., x_6 is a rectangular cartesian system of coordinates. The reason for the factor $\sqrt{2}$ is the following: under a rotation of physical coordinates σ_{ij} behaves like a second order cartesian tensor, the factor $\sqrt{2}$ is necessary in order that x_α , ($\alpha = 1,2,...6$), should transform like a cartesian vector.

The relations (1) and (2) may be compactly written as follows:

$$x_\alpha = B_{\alpha,ij}\ \sigma_{ij} \qquad \text{(A-3)}$$

in which $B_{\alpha,ij}$ are a set of 3x3 matrices whose definitions are:

$$B_{1,ij} = \frac{1}{2\sqrt{3}}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad B_{2,ij} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad B_{3,ij} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{4,ij} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad B_{5,ij} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad B_{6,ij} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{A-4}$$

Since all formulas concerning plastic strains do not involve the x_6 coordinate this coordinate will be dropped and the range of summation on the Greek indices will hereafter be understood to run from 1 to 5. The quantities $B_{\alpha,ij}$ satisfy the following identities which are easily verified.

$$B_{\alpha,ij} = B_{\alpha,ji}$$

$$B_{\alpha,ij}\, B_{\beta,ij} = \tfrac{1}{2}\, \delta_{\alpha\beta} \tag{A-5}$$

$$B_{\alpha,ii} = 0$$

$$B_{\alpha,ij}\, B_{\alpha,k\ell} = \frac{1}{4}\left(\delta_{ik}\, \delta_{j\ell} + \delta_{i\ell}\delta_{jk} - \frac{2}{3}\, \delta_{ij}\, \delta_{k\ell}\right).$$

The components of x_α depend only on the stress deviator as is shown directly by the following:

$$x_\alpha = B_{\alpha,ij}\, \sigma_{ij} = B_{\alpha,ij}\,(s_{ij} + s\,\delta_{ij}) = B_{\alpha,ij}\, s_{ij}$$

$$+ s\, B_{\alpha,ii} = B_{\alpha,ij}\, s_{ij} \tag{A-6}$$

Equation 6 may be solved for the components of the deviator in terms of x_α ; the result is:

$$s_{ij} = 2\, B_{\alpha,ij}\, x_\alpha \tag{A-7}$$

From (7) and (5) it follows:

$$J_2 = \tfrac{1}{2}\, s_{ij} s_{ij} = 2\, B_{\alpha,ij} B_{\beta,ij}\, x_\alpha x_\beta = \delta_{\alpha\beta} x_\alpha x_\beta = x_\alpha x_\alpha \tag{A-8}$$

that is the length of a stress vector x_α is $\sqrt{J_2}$.

Analogous to the stress vector a plastic strain vector is defined as follows:

$$e_\alpha = 2\, B_{\alpha,ij}\, \varepsilon_{ij}^{p} \tag{A-9}$$

The stress-strain relation for a linear stress theory of plasticity reads as follows in the present notation:

$$de_\alpha = G(f)\, \frac{\partial f}{\partial x_\alpha}\, df \tag{A-10}$$

as may easily be verified. The tensor character of this relation is preserved in the present representation and as before the plastic strain increment is normal to the loading surface.

<u>A coordinate system for isotropic materials</u>.

The geometrical representation of J_2 as given in the last section is quite simple but what about J_3? Explicitly J_3 is:

$$J_3 = \tfrac{1}{3} s_{ij} s_{jk} s_{ki} = \frac{2}{3\sqrt{3}}\, x_1^3 - \frac{2}{\sqrt{3}}\, x_1 (x_2^2 + x_4^2) + \frac{1}{\sqrt{3}}\, x_1 (x_3^2 + x_5^2) + x_2 (x_3^2 - x_5^2) + 2 x_3 x_4 x_5 \tag{A-11}$$

which is a rather awkward expression. There is a simple representation of J_3 in case $s_{ij} = 0$, $i \neq j$, in which case $x_3 = x_4 = x_5 = 0$. If polar coordinates are introduced into the 1,2 plane letting:

$$
\begin{aligned}
x_1 &= \rho \cos \Theta \\
x_2 &= \rho \sin \Theta
\end{aligned}
\qquad \text{(A-12)}
$$

then we find:

$$J_2 = \rho^2 \; ; \; J_3 = \frac{2\rho^3}{3\sqrt{3}} \cos 3\Theta \qquad \text{(A-13)}$$

The corresponding values of s_{ij} are:

$$
\begin{aligned}
s_{11} &= \frac{2\rho}{\sqrt{3}} \cos \Theta \\
s_{22} &= -\frac{2\rho}{\sqrt{3}} \cos \left(\Theta + \frac{\pi}{3}\right) \\
s_{33} &= -\frac{2\rho}{\sqrt{3}} \cos \left(\Theta - \frac{\pi}{3}\right)
\end{aligned}
\qquad \text{(A-14)}
$$

The reason for introducing J_2 and J_3 is of course that they are invariant under rotations of the physical coordinates and hence play a prominent role in the description of initially isotropic materials. We would like to construct a coordinate system in stress space convenient for such a description. In order to gain a little insight into the problem we consider the effect on the vector x_α of subjecting the corresponding tensor s_{ij} to a rotation in physical space. As is easily checked using (5) the new vector

x_α' corresponding to the new tensor s'_{ij} is related to the vector x_α by a rigid body rotation in stress space. That is to say

$$x_\alpha' = A_{\alpha\beta} \, x_\beta \tag{A-15}$$

where $A_{\alpha\beta}$ is an orthogonal matrix. If a_{ij} is the rotation tensor in physical space then $A_{\alpha\beta}$ is given by:

$$A_{\alpha\beta} = 2 \, a_{ik} \, a_{j\ell} \, B_{\alpha,ij} \, B_{\beta,k\ell} \tag{A-16}$$

Thus to every rotation in physical space there corresponds a rotation in stress space. The correspondence between rotations however is not one to one. There is a ten parameter group of rotations in the five dimensional stress space but only a three parameter sub-group of them corresponds to the three parameter group of rotations in physical space. The somewhat messy expression (11) is of course an invariant of this sub-group.

The correspondence between rotations in physical space and a certain group of rotations in stress space suggests at least one way in which a coordinate system especially adapted to the description of isotropic materials may be defined. We first note that for a given stress tensor the coordinate system in physical space can always be rotated into a system of principal axes for that stress tensor. This means that in stress space an arbitrary vector x_α can always be rotated into the 1,2 plane by a rotation belonging to the above mentioned sub-group. This can be accomplished in six different ways corresponding to the six choices of right handed principal axes in physical space.

This means that in stress space an arbitrary vector x_α can be rotated into any one of six equal sectors in the 1,2 plane by a rotation belonging to our sub-group. Thus the stress vectors x_α in the sector of the 1,2 plane $0 \leq \Theta \leq \frac{\pi}{3}$ are representative of all vectors in stress space through these rotations. The coordinate system is constructed as follows. Choose a vector in the 1,2 plane given by its length ρ and the angle Θ_4 it makes with the x_1 axis where $0 \leq \Theta_4 \leq \frac{\pi}{3}$. Next apply an arbitrary rotation belonging to our sub-group which will involve three parameters, call them Θ_1, Θ_2 and Θ_3. The resulting vector:

$$x_\alpha = x_\alpha (\rho, \Theta_1, \Theta_2, \Theta_3, \Theta_4) \qquad \text{(A-17)}$$

can be any vector in stress space so (16) can be regarded as the equations defining a transformation from x coordinates to ρ, Θ coordinates. With suitable restrictions on the range of Θ_1, Θ_2 and Θ_3 the new coordinate system covers stress space once and only once. Obviously the values of J_2 and J_3 are the same for any two stress states which have the same ρ and Θ_4 coordinates because these quantities are unaffected by rotations. Expressions for these invariants are still given by (13) where we take Θ to be Θ_4. The principal deviatoric stresses in the order $s_1 \geq s_2 \geq s_3$ are given by (14) when Θ_4 is restricted to the range $0 \leq \Theta_4 \leq \frac{\pi}{3}$. As an example of the use of this coordinate system note that the equation of the initial yield surface of an isotropic material must be of the form:

$$\rho = P\ (\Theta_4) \tag{A-18}$$

The explicit form of (17) may be obtained straightforwardly but the calculations are somewhat tedious so only the results will be given here. First let:

$$x_\alpha = \rho\ \xi_\alpha \tag{A-19}$$

then $\xi_\alpha = \xi_\alpha\ (\Theta_p)$, $(p = 1,2,3,4)$, defines a system of coordinates on the unit sphere in stress space. Expressions for ξ_α are as follows:

$$\xi_1 = \frac{1}{4}\ (1+3\cos 2\Theta_2)\cos\Theta_4 + \frac{\sqrt{3}}{4}\ (1-\cos 2\Theta_2)\cos 2\Theta_3 \sin\Theta_4$$

$$\xi_2 = \frac{\sqrt{3}}{4}(1-\cos 2\Theta_2)\cos 2\Theta_1 \cos\Theta_4 + \frac{1}{4}(3$$
$$+\cos 2\Theta_2)\cos 2\Theta_1 \cos 2\Theta_3 \sin\Theta_4$$
$$-\cos\Theta_2 \sin 2\Theta_1 \sin 2\Theta_3 \sin\Theta_4$$

$$\xi_3 = \frac{\sqrt{3}}{2}\sin 2\Theta_2 \cos\Theta_1 \cos\Theta_4 - \frac{1}{2}\sin 2\Theta_2 \cos\Theta_1 \cos 2\Theta_3 \sin\Theta_4$$
$$+\sin\Theta_2 \sin\Theta_1 \sin 2\Theta_3 \sin\Theta_4$$

$$\xi_4 = \frac{\sqrt{3}}{4}\ (1-\cos 2\Theta_2)\sin 2\Theta_1 \cos\Theta_4 + \frac{1}{4}(3$$
$$+\cos 2\Theta_2)\sin 2\Theta_1 \cos 2\Theta_3 \sin\Theta_4$$
$$+\cos\Theta_2 \cos 2\Theta_1 \sin 2\Theta_3 \sin\Theta_4$$

$$\xi_5 = \frac{\sqrt{3}}{2}\sin 2\Theta_2 \sin\Theta_1 \cos\Theta_4 - \frac{1}{2}\sin 2\Theta_2 \sin\Theta_1 \cos 2\Theta_3 \sin\Theta_4$$
$$-\sin\Theta_2 \cos\Theta_1 \sin 2\Theta_3 \sin\Theta_4 \tag{A-20}$$

The ranges of the angles θ_p are as follows:

$$0 \leq \theta_1 < 2\pi \qquad 0 \leq \theta_3 < \pi$$

$$0 \leq \theta_2 \leq \pi \qquad 0 \leq \theta_4 \leq \frac{\pi}{3} \tag{A-21}$$

The angles θ_p are suitable as a system of coordinates on the initial yield surface of an isotropic material which is given by:

$$x_\alpha = P(\theta_4)\, \xi_\alpha\, (\theta_p) \tag{A-22}$$

The element of area on this surface is given by:

$$da = 2P^4\, [1 + (P'/P)^2]^{\frac{1}{2}} \sin\theta_2 \sin 3\theta_4\, d\theta_1\, d\theta_2\, d\theta_3\, d\theta_4 \tag{A-23}$$

The unit normal vector to the surface is given by:

$$n_\alpha = \frac{\xi_\alpha - \frac{P'}{P}\frac{\partial \xi_\alpha}{\partial \theta_4}}{[1 + (P'/P)^2]^{\frac{1}{2}}} \tag{A-24}$$

Yield surfaces.

In this section a few well known facts concerning the initial yield surface for an isotropic material are restated in terms of the system of coordinates given in the last section. Such a yield surface is of course invariant under those rotations in stress space which correspond to rotations in physical space. Obviously the yield boundary in the 1,2 plane determines the whole surface, in fact the arc of the boundary which lies in the sector $0 \leq \theta_4 \leq \frac{\pi}{3}$ is sufficient to determine the whole surface. Three of the axes of the permissable rotations lie in the 1,2 plane; one is the x_1 axis and the other two make angles of $\frac{\pi}{3}$ with the x_1 axis. These three lines must therefore be axes of symmetry of the yield boundary.[1]

The yield surface must also be convex.[2] It is easy to show that if the yield boundary in the 1,2 plane is convex then this condition is satisfied. Let us assume that $P(\theta_4)$ has two continuous derivatives. The condition of convexity is explicitly:

$$P^2 + 2P'^2 - PP'' \geq 0 \qquad 0 \leq \theta_4 \leq \frac{\pi}{3}$$
$$P'(0) \leq 0 \leq P'(\tfrac{\pi}{3}) \tag{A-25}$$

The equality sign holds in the first inequality for a straight line. The Tresca yield surface is a special case of this latter possibility. If the yield surface is to be smooth the yield boundary in the 1,2 plane must meet the rays $\theta_4 = 0$ and $\theta_4 = \frac{\pi}{3}$ at right angles, this will be true if:

$$P = R(\cos 3\theta_4) \tag{A-26}$$

where R is regular for $0 \leq \theta_4 \leq \frac{\pi}{3}$. If in addition we are to have point symmetry about the origin then

$$P = R(\cos^2 3\theta_4) \tag{A-27}$$

---

[1] All members of the sub-group except the identity have one fixed axis.

[2] By this we mean that no part of a straight line segment joining any two points of the yield surface falls outside the yield surface.

The fact that the yield surfaces (initial or subsequent) considered in this paper are always convex is intuitively obvious considering the fact that they are all the "innermost" envelope of a set of planes.

Some additional spherical coordinate systems.

Several orthogonal systems of coordinates on the unit sphere in five dimensions may be manufactured as in the following example. Let the projection of an arbitrary unit vector on the (x_1,x_2) subspace be of length $\cos\Theta_2$ so that the length of the projection on the orthogonal complement (x_3,x_4,x_5) is $\sin\Theta_2$. Let the projections on the x_1 and x_2 axes be:

$$\begin{aligned}\xi_1 &= \cos\Theta_2 \cos\Theta_1 \\ \xi_2 &= \cos\Theta_2 \sin\Theta_1\end{aligned} \tag{A-28}$$

as in plane polar coordinates. In the (x_3,x_4,x_5) subspace the projections on the axes are made as in ordinary spherical polar coordinates to give:

$$\begin{aligned}\xi_3 &= \sin\Theta_2 \sin\Theta_3 \cos\Theta_4 \\ \xi_4 &= \sin\Theta_2 \cos\Theta_3 \\ \xi_5 &= \sin\Theta_2 \sin\Theta_3 \sin\Theta_4\end{aligned} \tag{A-28$'$}$$

The ranges of the variables are:

$$0 \leq \theta_1 < 2\pi \qquad 0 \leq \theta_3 \leq \pi$$
$$0 \leq \theta_2 \leq \frac{\pi}{2} \qquad 0 \leq \theta_4 < 2\pi \tag{A-29}$$

The element of area on the unit sphere is given by:

$$da = \cos\theta_2 \sin^2\theta_2 \sin\theta_3 \, d\theta_1 \, d\theta_2 \, d\theta_3 \, d\theta_4 \tag{A-30}$$

J_3 is given by:

$$\frac{1}{\rho^3} J_3 = \frac{2}{3\sqrt{3}} \cos^3\theta_2 \cos 3\theta_1 - \frac{1}{2\sqrt{3}} \cos\theta_1 \cos\theta_2 \sin^2\theta_2 (1 - 3\cos 2\theta_3) + \sin^2\theta_2 \sin^2\theta_3 (\sin\theta_1 \cos\theta_2 \cos 2\theta_4 + \sin\theta_2 \cos\theta_3 \sin 2\theta_4) \tag{A-31}$$

Another system of coordinates is:

$$\xi_1 = \cos\theta_1$$
$$\xi_2 = \sin\theta_1 \sin\theta_2 \cos\theta_3 \qquad 0 \leq \theta_1 \leq \pi$$
$$\xi_3 = \sin\theta_1 \cos\theta_2 \cos\theta_4 \qquad 0 \leq \theta_2 \leq \frac{\pi}{2}$$
$$\xi_4 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \qquad 0 \leq \theta_3 < 2\pi$$
$$\xi_5 = \sin\theta_1 \cos\theta_2 \sin\theta_4 \qquad 0 \leq \theta_4 < 2\pi$$

$$da = \sin^3\theta_1 \sin\theta_2 \cos\theta_2 \, d\theta_1 \, d\theta_2 \, d\theta_3 \, d\theta_4$$

$$\frac{3\sqrt{3}}{2\rho^3} J_3 = \cos 3\theta_1 + \frac{3}{2} \sin^2\theta_1 \cos^2\theta_2 \left[3\cos\theta_1 + \sqrt{3} \sin\theta_1 \sin\theta_2 \cos(2\theta_4 - \theta_3)\right] . \tag{A-32}$$

Appendix B

Solving for H(f).

Make the x_α coordinates dimensionless by letting:

$$x_\alpha = \frac{1}{k} B_{\alpha,ij} \sigma_{ij} \qquad \text{(B-1)}$$

where k is the yield stress in pure shear. Let the yield surface be $\rho = 1$ and treat all planes equally, then the stress-strain relation is:

$$e_\alpha = \int_s H(f)\, \xi_\alpha \, da \qquad \text{(B-2)}$$

The function H has to be determined in terms of experimental information, say a stress-strain curve in tension. Let this be given by:

$$\sigma_x = E_s \, \varepsilon_x \qquad \text{(B-3)}$$

where E_s is the secant modulus. The plastic part of the strain is given by:

$$\varepsilon_x^{\,p} = \left(\frac{1}{E_s} - \frac{1}{E}\right) \sigma_x \qquad \text{(B-4)}$$

For a simple tension test $\sigma_x = \sqrt{3}\, x_1 = \sqrt{3}\, \rho$ and $e_1 = \sqrt{3}\, \varepsilon_x^{\,p}$ so (4) becomes:

$$e_1 = 3\left(\frac{1}{E_s} - \frac{1}{E}\right) \rho = 2F(\rho) \qquad \text{(B-5)}$$

In the polar coordinate system defined by eqs. (A-32) the distance of a plane from the origin for the present path is:

$$f = \rho \cos \theta_1 \qquad \text{(B-6)}$$

(see fig. 11) and (2) becomes:

$$2F(\rho)$$

$$= \int_0^{\bar{\theta}_1} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{2\pi} H(\rho \cos \theta_1) \cos \theta_1 \sin^3\theta_1 \sin \theta_2 \cos \theta_2 d\theta_4 d\theta_3 d\theta_2 d\theta_1 \qquad \text{(B-7)}$$

$$= 2\pi^2 \int_0^{\bar{\theta}_1} H(\rho \cos \theta_1) \cos \theta_1 \sin^3\theta_1 \, d\theta_1 \qquad \text{(B-8)}$$

Let $\eta = \rho \cos \theta_1$, then (8) becomes:

$$\rho^4 F(\rho) = \pi^2 \int_1^{\rho} H(\eta)\, \eta \, (\rho^2 - \eta^2)\, d\eta \qquad \text{(B-9)}$$

This may be solved for H by mere differentiation to give:

$$H(\rho) = [(\rho^4 F)'/\rho]'/2\pi^2\rho \qquad \text{(B-10)}$$

<u>Reduction to loading lines</u>.

For loading in the x_1, x_2 plane it is convenient to reformulate the problem in terms of loading lines in that plane. In the present case of course, because of symmetry, exactly the same analysis applies to say the x_1, x_3 plane or any other flat two dimensional subspace of stress space which includes the origin. The solution in these other cases may be obtained from that in the x_1, x_2 plane by a transformation of coordinates or even by merely relabeling the axes.

In the coordinate system (A-28) (2) becomes:

$$e_1 = \int_0^{2\pi} \overline{H}(\overline{f}) \cos \theta_1 \, d\theta_1 \qquad \text{(B-11)}$$

$$e_2 = \int_0^{2\pi} \overline{H}(\overline{f}) \sin \theta_1 \, d\theta_1 \qquad \text{(B-12)}$$

where:

$$\overline{H}(\overline{f}) = \pi \int_0^{\frac{\pi}{2}} H(\overline{f} \cos \theta_2) \sin^2 2\theta_2 \, d\theta_2 \qquad \text{(B-13)}$$

and where $\overline{f}$ is the distance of a loading line from the origin in the x_1, x_2 plane. By using (10) and after considerable reduction (13) becomes:

$$\overline{H}(\rho) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} [\rho \cos \theta \, F'(\rho \cos \theta) + F(\rho \cos \theta)] \, d\theta \qquad \text{(B-14)}$$

Formula (14) may also be obtained directly using (11) and the loading line approach without resorting to (10) and then if so desired H may be obtained by inverting (13), in other words it is possible to return to loading planes after having determined $\overline{H}$ by the use of loading lines. However if one is only interested in loading in the x_1, x_2 plane this is unnecessary.

In terms of loading lines the plastic strain resulting from a simple tension test is given by (11):

$$2F(\rho) = \int_{-\overline{\theta}}^{\overline{\theta}} \overline{H}(\rho \cos \theta) \cos \theta \, d\theta \qquad \text{(B-15)}$$

where $\rho \cos \overline{\theta} = 1$. Figure 11 serves as an illustration in this case also but with a different interpretation than before. Equation (15) may as well be written:

$$F(\rho) = \int_0^{\frac{\pi}{2}} \overline{H} (\rho \cos \theta) \cos \theta \, d\theta \qquad \text{(B-16)}$$

because $\overline{H}(\rho) = 0$ for $\rho \leq 1$. Equation (16) is not quite as simple an integral equation to solve as (8). (See [18]) However equations like (13) and (16) may be solved by first making a change of variable. Let

$$r = \rho^2 \; ; \; s = \rho^2 \cos^2 \theta$$

then (16) becomes:

$$2 \, r^{\frac{1}{2}} \, F(r^{\frac{1}{2}}) = \int_0^r \frac{\overline{H}(s^{\frac{1}{2}}) \, ds}{(r - s)^{\frac{1}{2}}} \tag{B-17}$$

This is the familiar Abel's eq. and the solution is:

$$\overline{H}(r^{\frac{1}{2}}) = \frac{2}{\pi} \frac{d}{dr} \int_0^r \frac{s^{\frac{1}{2}} \, F(s^{\frac{1}{2}}) \, ds}{(r - s)^{\frac{1}{2}}}$$

$$= \frac{2}{\pi} \int_0^r \frac{d\left[s^{\frac{1}{2}} \, F(s^{\frac{1}{2}})\right]}{(r - s)^{\frac{1}{2}}} \tag{B-18}$$

Transforming back again gives (14).

<u>Broken line loading path</u>.

The plastic strain resulting from any loading path in the x_1, x_2 plane may now be written down in closed form in terms of $\overline{H}$, the function $\overline{H}$ having been computed from the stress-strain curve using (14). If however one wished to compute the strain for a large number of broken line loading paths it would be more economical to have a specific formula for that case. This formula has been worked out. Suppose the resulting yield boundary is as shown in fig. 12 (whatever the loading path), then the plastic strains are given by:

$$e_1 = F(\rho_o) + \cos(\alpha+\beta)F\ (\rho_1) - \text{sign}\ \alpha[C(\alpha,\gamma) - F(\rho_o)]$$

$$- \text{sign}\ \beta\ [C(\beta,\gamma) - F(\rho_1)\ ]\cos\ (\alpha+\beta) \qquad \text{(B-19)}$$

$$e_2 = \sin\ (\alpha+\beta)\ F(\rho_1) - \text{sign}\ \beta\ [C(\beta,\gamma) - F(\rho_1)]\ \sin\ (\alpha+\beta) \text{(B-20)}$$

where:

$$C(\alpha,\gamma) = \frac{2}{\pi} \sin |\alpha| \int_o^{\frac{\pi}{2}} \frac{F(\cos\ \Theta/\cos\ \gamma)\ d\Theta}{1 - \cos^2\alpha\cos^2\Theta} \qquad \text{(B-21)}$$

Note:

$$\lim_{\alpha \to o} C(\alpha,\gamma) = F(1/\cos\ \gamma) \qquad \text{(B-22)}$$

Appendix C

Uniform expansion.

In the coordinate system (A-32) eq.(31) with $\mu = 0$ becomes:

$$r_2 = \lambda \int_0^{\bar{\Theta}_1} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{2\pi} (\rho \cos \Theta_1 - r_2 - 1) \sin^3\Theta_1 \sin\Theta_2 \cos\Theta_2 \, d\Theta_4 \, d\Theta_3 \, d\Theta_2 \, d\Theta_1 \tag{C-1}$$

(see Fig.9). Carrying out the integrations on $\Theta_2, \Theta_3, \Theta_4$ and dropping the subscript on Θ_1 gives:

$$r_2 = 2\lambda\pi^2 \int_0^{\bar{\Theta}} (\rho \cos \Theta - 1 - r_2) \sin^3\Theta \, d\Theta \tag{C-2}$$

where $\rho \cos \bar{\Theta} = 1 + r_2$. Equation (2) may be written:

$$\rho \cos \bar{\Theta} - 1 = 2\lambda\pi^2\rho \int_0^{\bar{\Theta}} (\cos \Theta - \cos \bar{\Theta}) \sin^3\Theta \, d\Theta \tag{C-3}$$

One more integration gives:

$$\rho \cos \bar{\Theta} - 1 = \frac{1}{6} \lambda\pi^2\rho \, (3 - 8 \cos \bar{\Theta} + 6 \cos^2\bar{\Theta} - \cos^4\Theta) \tag{C-4}$$

The r.h.s. of (4) has the same sign as λ and is a monotone function of $\bar{\Theta}$ for $0 \leq \bar{\Theta} \leq \frac{\pi}{2}$ as is evident from (3). Solving (4) for ρ gives:

$$\rho = \left[\cos \bar{\Theta} - \frac{1}{6} \lambda\pi^2(3 - 8 \cos \bar{\Theta} + 6 \cos^2\bar{\Theta} - \cos^4\bar{\Theta} \right]^{-1} \tag{C-5}$$

For $0 < \lambda < \infty$, ρ increases steadily with $\bar{\Theta}$ and becomes infinite sometime before $\bar{\Theta}$ reaches $\pi/2$. However for $\lambda < 0$ no matter how small in absolute value the denominator in (5) is always positive for $0 \leq \overline{\Theta} \leq \frac{\pi}{2}$. Thus ρ does not become infinite as $\overline{\Theta}$ increases from 0 but reaches some maximum value and then decreases until $\overline{\Theta}$ reaches $\pi/2$ at which time the elastic region disappears altogether. The non-existence of solutions for arbitrarily large ρ and the possibility of two solutions for some values of ρ is untenable, therefore negative values of λ must be excluded. For ρ fixed and $\lambda \rightarrow \infty$ it is evident from (5) that $\overline{\Theta} \rightarrow 0$, in other words the corner disappears as $\lambda \rightarrow \infty$.

It is possible to depart from a radial loading path by turning the path through a sufficiently small angle without unloading any planes which have already been loaded. Obviously all planes which are continuing to load must pass through the loading point and if no plane which has ever been loaded has subsequently unloaded then all planes which have been loaded pass through the loading point. In this case the solution is that just given for radial loading which is unique for $\lambda \geq 0$. It follows that the solutions for nearly radial loading are also unique. The question of whether or not there is a unique solution for non-nearly-radial loading paths is left open. The permissable angle of turning for a nearly radial loading path is determined in the following.

Suppose the loading has been along the x_1 axis up to some point P and then a small increment of load is added to reach the point P' off the x_1 axis (see fig. 13). As shown in the figure, γ has been the angle of turning. The line marked t is the trace of a single plane and this plane before all others will begin to unload if γ is too large. In the figure t' is the trace of this same plane after the increment of load has been added. As shown this plane has remained in contact with both the loading point and the expanding spherical portion of the yield surface; therefore it has been neutrally loaded and γ is the maximum permissable angle of turning. From the figure half the angle of the corner at P' is given in two ways:

$$\frac{\pi}{2} - \overline{\Theta} - d\overline{\Theta} = \frac{\pi}{2} - \overline{\Theta} - d\alpha$$

$$\therefore \quad d\overline{\Theta} = d\alpha \qquad \text{(C-6)}$$

Also $\qquad d\rho = ds \cos \gamma \qquad$ (C-7)

and $\qquad \rho d\alpha = ds \sin \gamma \qquad$ (C-8)

From (6-8) it follows $\cot \gamma = \frac{1}{\rho} \frac{d\rho}{d\overline{\Theta}}$ (C-9)

Using (5), after some manipulation (9) becomes:

$$\cot \gamma = \rho \sin \overline{\Theta} + 4 \cot \tfrac{1}{2} \overline{\Theta} \, (\rho \cos \overline{\Theta} - 1) \, \frac{2+\cos \overline{\Theta}}{3+\cos \overline{\Theta}} \qquad \text{(C-10)}$$

For fixed ρ as $\lambda \to \infty$, $\overline{\Theta} \to 0$ $\therefore$ $\gamma \to 0$ and the permissible angle of turning vanishes. As $\lambda \to 0$, $\rho \cos \overline{\Theta} \to 1$ and $\cot \gamma \to \tan \overline{\Theta}$ as it should. The permissable angle of turning decreases as ρ increases for fixed $\lambda > 0$ and is always less than half the angle of the corner. Of course no part of the nearly radial path need be radial as was assumed at the beginning of the argument, what is essential is that the loading path proceed into the cone beyond P with apex angle 2γ and axis OP.

Translation.

A procedure similar to that in the last section gives ρ as a function of $\bar{\Theta}$ (see fig. 14). The result is:

$$\rho \cos \bar{\Theta} = 1 + \frac{4}{15} \mu\pi^2 \sin^6 \frac{1}{2} \bar{\Theta}(3 \cos^2\bar{\Theta} + 9 \cos \bar{\Theta} + 8) \quad \text{(C-11)}$$

If $\mu \leq -15/4\pi^2$ there is the same sort of objectionable behavior as in the last section for $\lambda < 0$. In the present case also it is evident from (11) that $\bar{\Theta} \to 0$ as $\mu \to \infty$ for fixed ρ . The solutions for nearly radial loading paths will be unique if it can be shown that there is a unique solution for the case in which all planes that have ever been loaded pass through a given point, say P. When the path is exactly radial and $\mu > 0$ as in fig. 14 there is not much doubt about the uniqueness of the solution because of the symmetry of the situation. But what if the loading path had reached P by a curve slightly above the x_1 axis -- might not the yield surface look something like fig. 15 while still all planes that had ever been loaded pass through P? This can not be case for $\mu > 0$ according to the following argument. In the case shown in fig. 15 the configuration of loading planes is perfectly symmetrical about the axis O'P and thus so is the distribution of direct motion. The displacement vector A_α discussed in the text must be in the direction O'P for $\mu > 0$, but this would have displaced the spherical part of the yield surface downward instead of upward as it is shown in the figure. The only possible position for O' is on the line OP. The argument does not hold of course if $\mu < 0$, nor would it hold for other initial yield surfaces than spherical ones. However in the present case it is certain that the center of the spherical part of the yield surface must lie on the line OP if all planes which have ever been loaded pass through P. The half apex angle of the cone of total loading is given by the same formula as (9).

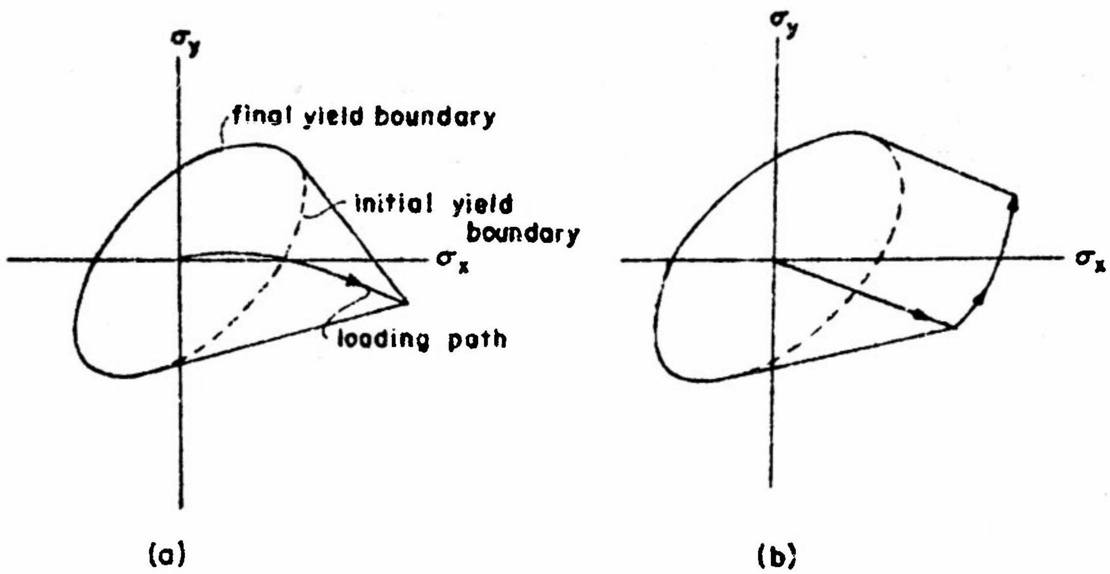

Fig. I. Corners appear in the yield boundary

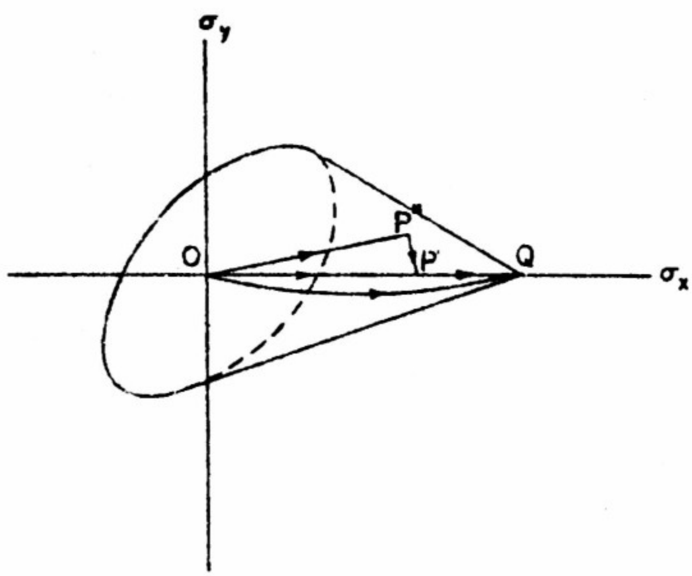

Fig. 2. Limited path independence

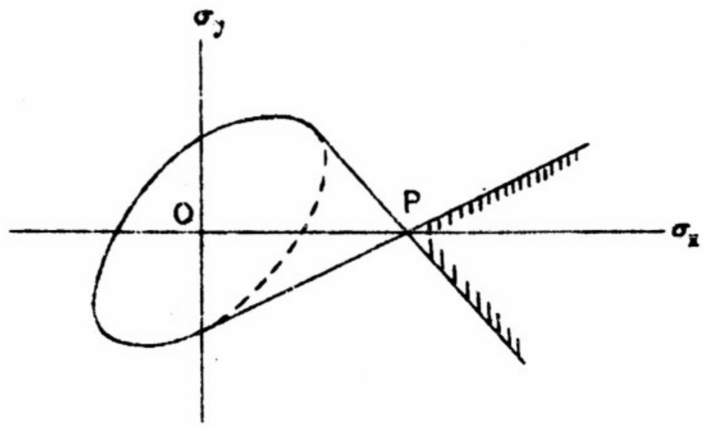

Fig. 3. Region of total loading

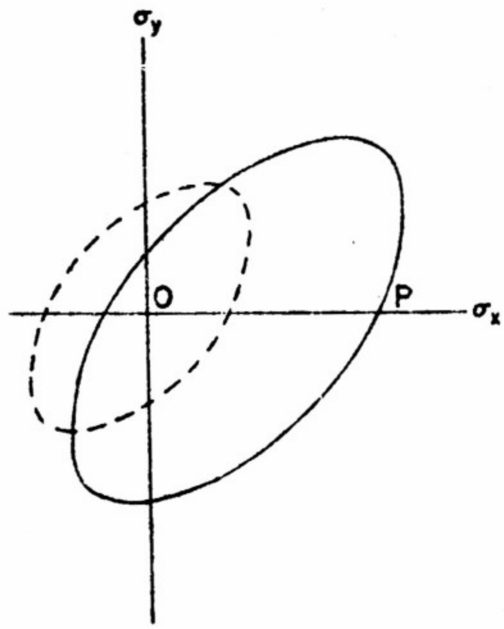

Fig. 4. Bauschinger effect

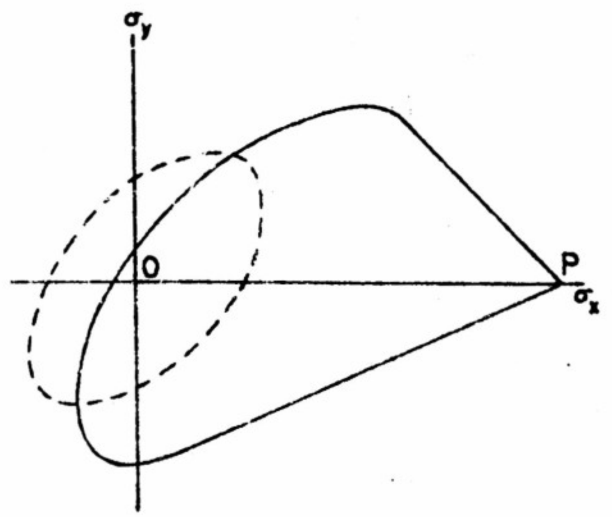

Fig. 5. Yield surface with a Bauschinger effect as envelope of plane loading surfaces

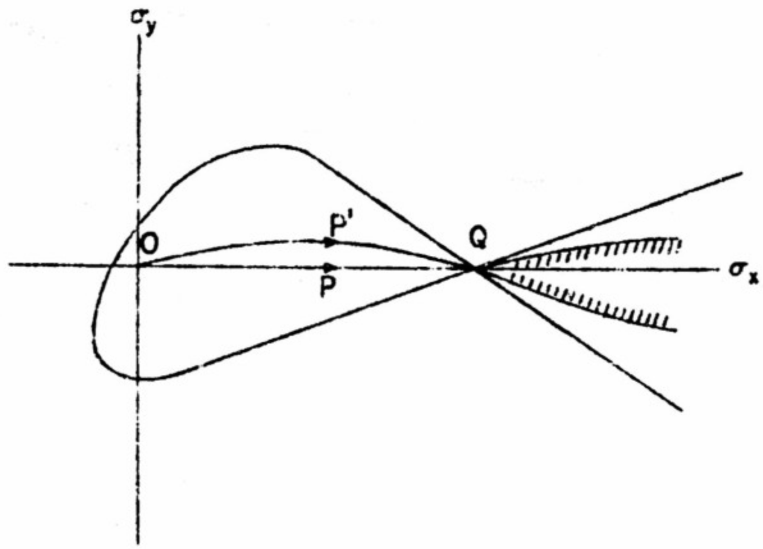

Fig. 6. Limited path independence with Bauschinger effect

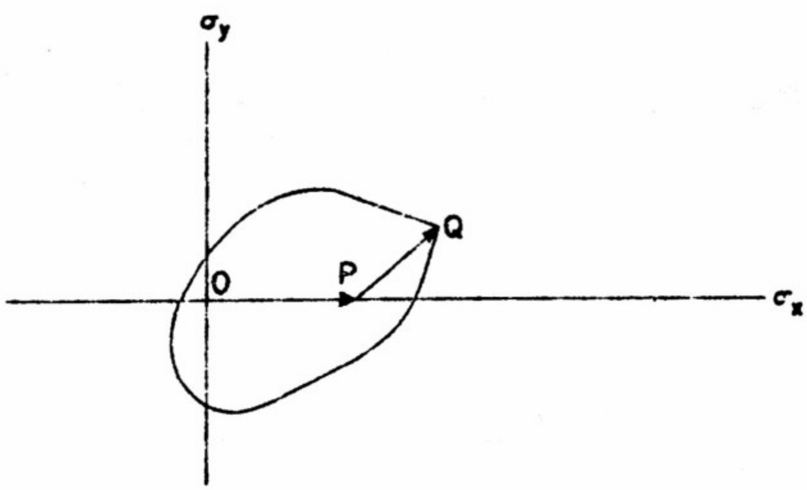

Fig. 7. Yield boundary after non nearly radial loading

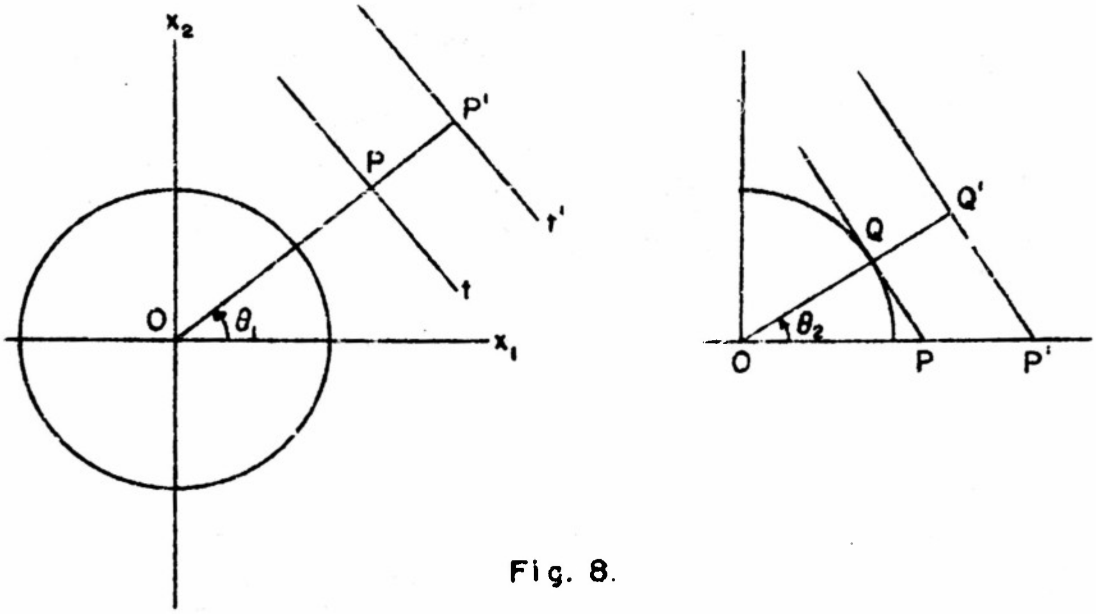

Fig. 8.

Motion of a plane related to the motion of its trace

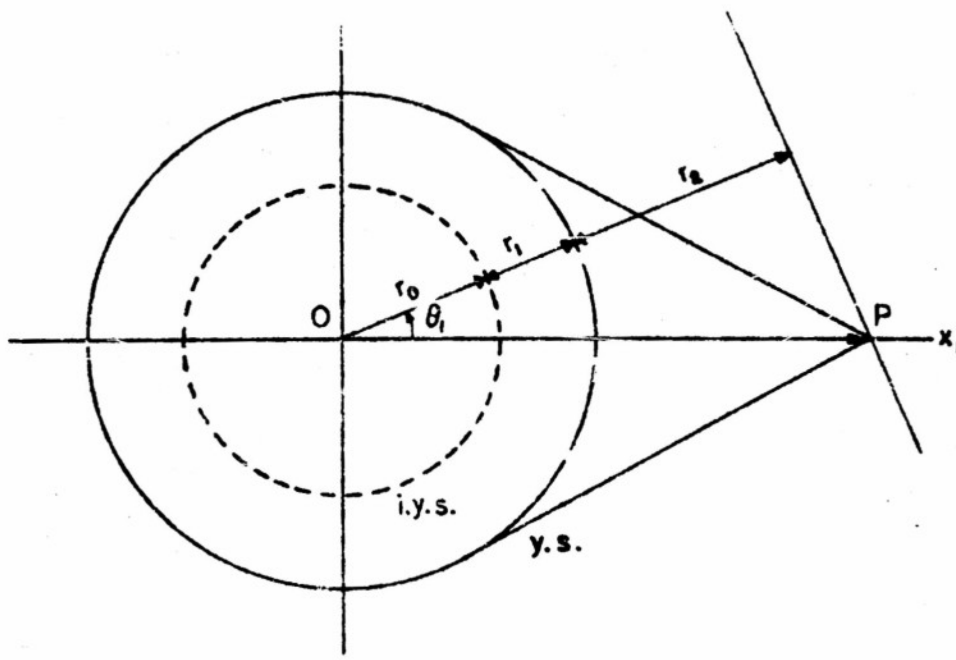

Fig. 9. Cross section of expanding yield surface

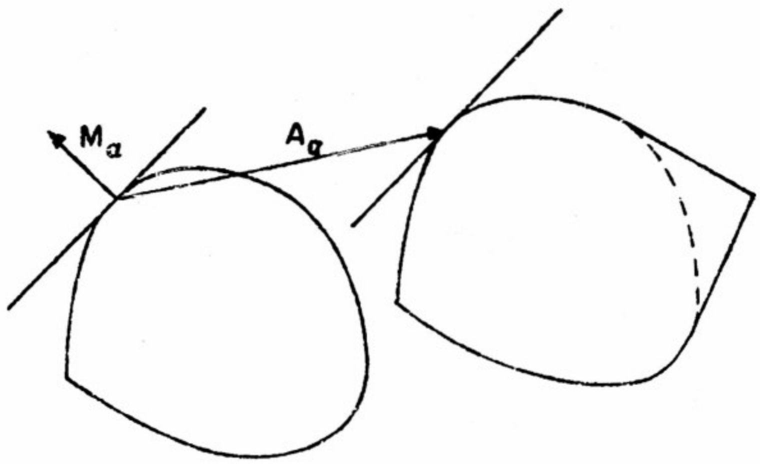

Fig. 10. Part of the yield surface displaced as a rigid body

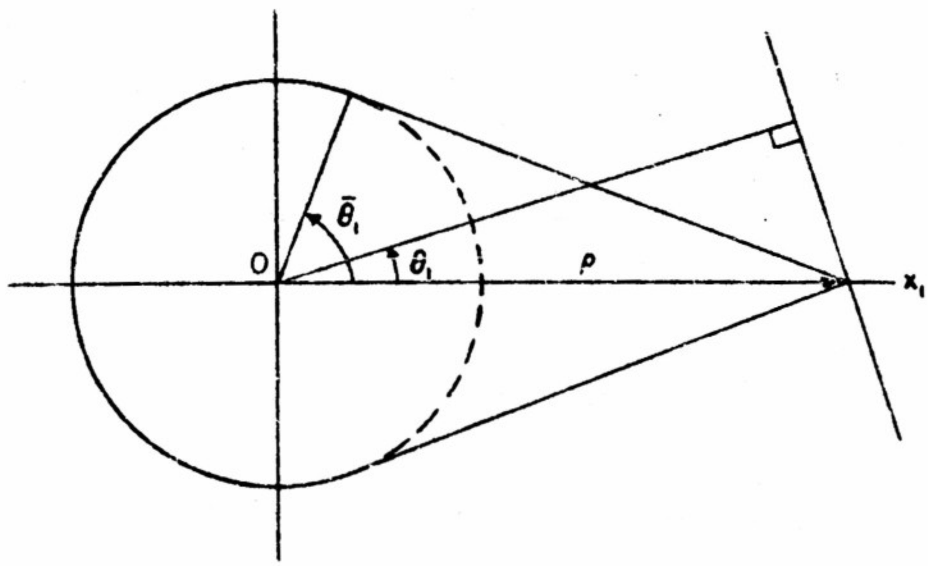

Fig. 11. Simple tension

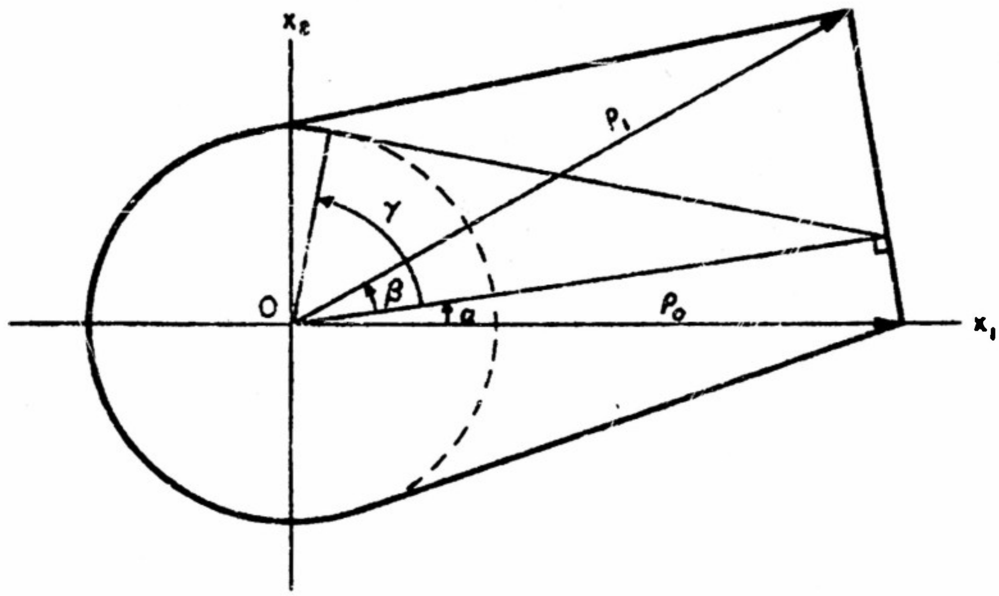

Fig. 12. Broken line loading path

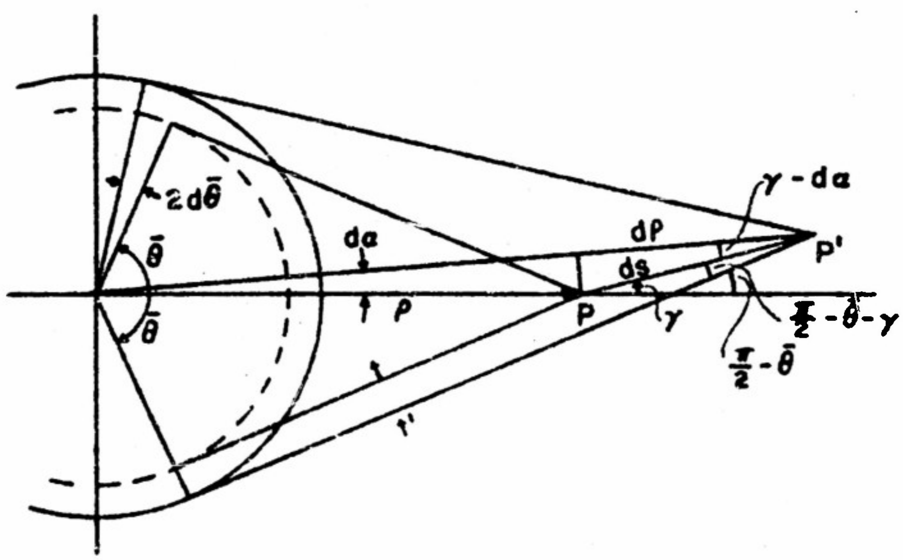

Fig. 13. Critical turning angle

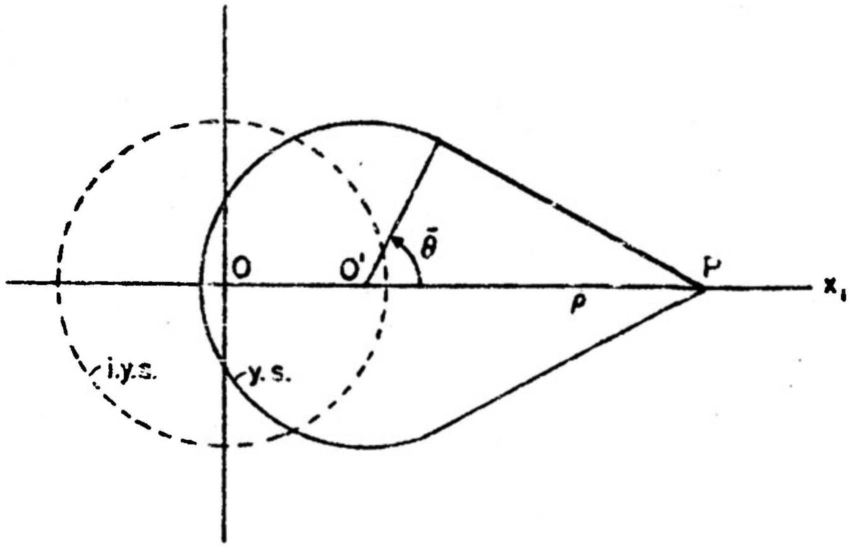

Fig. 14. Yield surface translated by radial loading

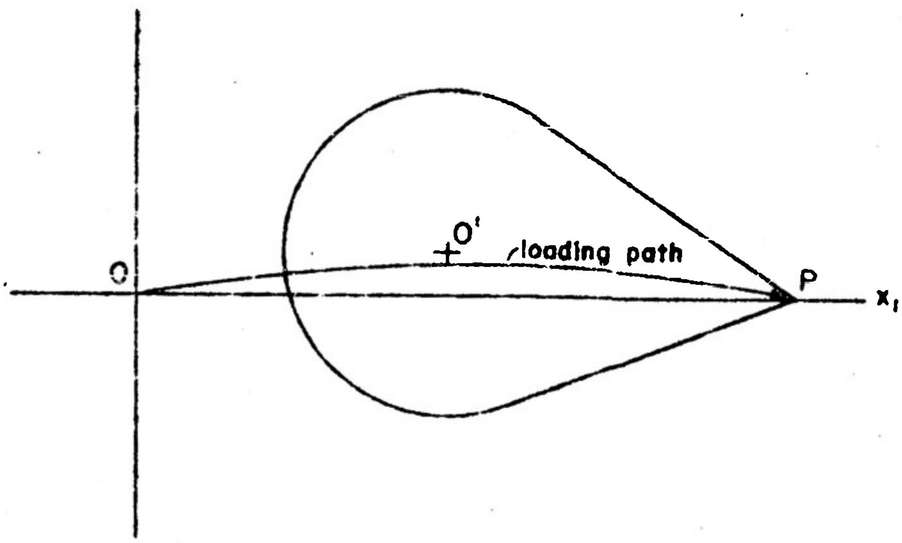

Fig. 15. Impossible case for nearly radial loading to P