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THE INFLUENCE OF SHEARING FORCES ON THE PLASTIC BENDING OF WIDE BEAMS

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THE INFLUENCE OF SHEARING FORCES ON THE PLASTIC BENDING OF WIDE BEAMS[1]

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ABSTRACT

The theory of the slip line field in plane plastic flow is used to obtain critical combinations of bending moment and shearing force for a wide beam of rectangular cross section.

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1. Introduction

In the limit analysis of plastic beams the influence of axial and shearing forces is usually neglected. Recently, the basic concepts of limit analysis were extended to members in which axial forces as well as bending moments must be taken into account [1,2]*. In the present paper the influence of the shearing forces on the plastic bending of a wide beam of rectangular cross section is investigated.

An exact solution of the problem would be rather difficult to obtain in the absence of general methods of plastic-elastic stress analysis. For a very wide beam, however, the problem becomes one in plane plastic flow. Critical states of loading may then be determined by the usual method [3,4] of finding stress and velocity fields that satisfy the equations of plane plastic flow and the boundary conditions of the problem.

2. Formulation of the problem

Consider a wide cantilever beam of constant rectangular cross section carrying the transverse load P per unit width at the free end, and denote the span of the beam by ℓ and its depth by 2h (Fig. 1). If this beams consists of a perfectly plastic-rigid material, a load of sufficient intensity will produce plastic flow in the neighborhood of the built-in end. If the ratio of the width to the depth of the beam is sufficitly large, this plastic flow can be treated as plane and the theory of the slip line field [3,4] can be used.

Let the lines $\alpha(x,y)$ = const. and $\beta(x,y)$ = const. be the slip lines, the direction of the maximum principal stress bisecting the angle formed by the directions of increasing α and β. Denote

* Numbers in square brackets refer to the Bibliography at the end of the paper.

the mean pressure by p, the yield stress in simple shear by k, the angle between the directions of increasing x and increasing α by φ, and the velocity components in the directions of increasing α and β by u and v, respectively. The theory of the slip line field requires that

$$\left.\begin{aligned} p + 2k\varphi &= \text{const.}, \\ du - v\,d\varphi &= 0, \\ dy - dx \tan\varphi &= 0, \end{aligned}\right\} \text{ along an } \alpha\text{-line (i.e. a line } \beta = \text{const.)} \tag{1}$$

and

$$\left.\begin{aligned} p - 2k\varphi &= \text{const.}, \\ dv + u\,d\varphi &= 0, \\ dy + dx \cot\varphi &= 0. \end{aligned}\right\} \text{ along a } \beta\text{-line (i.e. a line } \alpha = \text{const.)} \tag{2}$$

The stress and velocity fields must also satisfy the corresponding conditions:

a) the rate of plastic work must be non-negative at every part of the field;

b) it must be possible to extend the stress field from the plastically deforming into the rigid region in such a manner that the equations of equilibrium are satisfied and the yield condition is not violated.

The solution to the present problem must satisfy the following boundary conditions:

a) the stress components σ_y and τ_{xy} vanish on the stress-free surfaces AM and BL (Fig. 1);

b) the integral of the shearing stress τ_{xy} extended over the unit width of the end section LM equals the load P;

c) the velocity components u and v vanish at the built-in secion AB.

Note that the kinematic condition c) is rather stringent since it does not allow any change of shape or displacement of the section AB.

3. First type of slip line field

Figure 2 shows a first type of slip line field, which is found to be valid for $1.16 < \ell/h < 27.8$. This field is symmetrical with respect to the x axis; its upper half consists of a region of constant state (AEF), a fan (ADE) centered at A, and an isolated circular slip line arc CD. Such velocity fields with isolated circular slip lines were first discussed by Green [5] in his investigation of the flexure of notched bars in plane plastic flow.

The straight slip lines of the region AEF must meet the stress free surface AM at 45°. Throughout this region, the state of stress is specified by $\sigma_x = k$, $\sigma_y = \tau_{xy} = 0$, and the mean pressure is $p = -k$. Consequently, the α- and β-directions are as indicated in Fig. 2. The variation of pressure in the fan AED of α-lines is given by the first Eq. (2), and the variation of pressure along the circular α-line DC is given by the first Eq. (1). The terminal α-line AD of the fan and its circular continuation DC are determined from the condition that $p = 0$ at C. The fan angle θ_o is thus found to be

$$\theta_o = 8^\circ 10.6' \tag{3}$$

The velocity field associated with this shear line field is as follows. The material to the right of FEDC rotates as a rigid body about the center O of the arc CD. We will choose the angular velocity of this rotation as the unit of angular velocity. The material to the left of ADC remains at rest.

The velocities in the plastically deforming region are then determined from the condition that only the velocity component tangential to rigid-plastic interface is allowed a discontinuity, while the component normal to the interface must be continuous.

According to the second Eq.(2), the velocity component u has a constant value along a given α line of the fan ADE. Along the line AQ (Fig. 3), for instance, the value of u is obtained from the condition that there should be no discontinuity in u as we cross the plastic-rigid interface DE. Since the material below DE rotates with unit angular velocity about O, the intensity of u along AQ is given by the length of the perpendicular OQ from O onto AQ. Similarly, the value of u along the generic α line Q'Q" in the region of constant state AEF is given by the length of the perpendicular OQ" from O onto Q'Q".

With the notations of Fig. 3, we have along AQ:

$$u = r \cos \theta + b \sin \theta. \tag{4}$$

Since $d\varphi = - d\theta$ in the fan ADE, the second Eq.(2), Eq.(4) and the boundary condition $v = 0$ for $\theta = 0$ yield

$$v = - r \sin \theta + b(\cos \theta - 1). \tag{5}$$

In the region of constant state ADE, the velocity component v has the constant value obtained from (5) by replacing θ by θ_0. Thus the velocity field in this region is one of simple shear.

The rate of plastic work at a generic point of the described stress and velocity fields is readily determined and found to be positive.

The tractions transmitted across the line ADC must be equipollent to one half of the load P. With $\psi_o = \pi/4 - \theta_o$, this condition furnishes the following relations:

$$\frac{P}{2kh} = 2\,[\cos\psi_o - \psi_o \csc\psi_o]\,\frac{b}{h} + 2\psi_o \cot\psi_o - 1, \qquad (5)$$

$$\frac{P\ell}{2kh^2} = [-(\frac{b}{h})^2 + 2\,\frac{b}{h}\cos\psi_o\,][\,\psi_o \csc^2\psi_o - \cot\psi_o\,] + \psi_o \csc^2\psi_o + [1 - 2\psi_o \cot\psi_o\,]\cot\psi_o \qquad (6)$$

Equations (5) and (6) give P and ℓ/h as functions of the parameter b/h. Since negative values of this parameter are meaningless, Eqs.(5) and (6) cease to apply when b/h = 0. The corresponding value of ℓ/h is found to be 1.16.

4. <u>Second type of slip line field</u>.

For the critical value $\ell/h = 1.16$, the isolated circular slip lines of the field shown in Fig. 2 join the points A and B. For $\ell/h < 1.16$ an isolated circular slip line such as is shown in Fig. 4 may be considered. The values of P/(2kh) and ℓ/h for this field can be obtained from Eqs. (5) and (6) by setting b=0 and replacing ψ_o by ψ (see Fig. 4). Thus

$$\frac{P}{2kh} = 2\psi \cot\psi - 1 \qquad (7)$$

$$\frac{P\ell}{2kh^2} = \frac{\psi}{\sin^2\psi} + [1 - 2\psi\cot\psi\,]\cot\psi. \qquad (8)$$

The velocity field associated with this solution represents a rigid body rotation of the beam beyond the arc ACB about the center O of this arc (Fig. 4).

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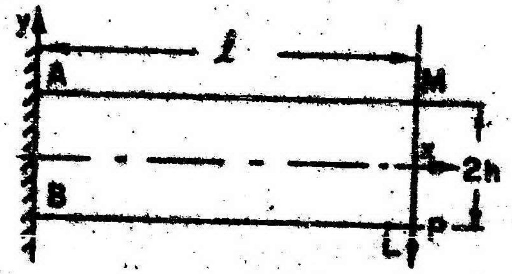

FIG. 1

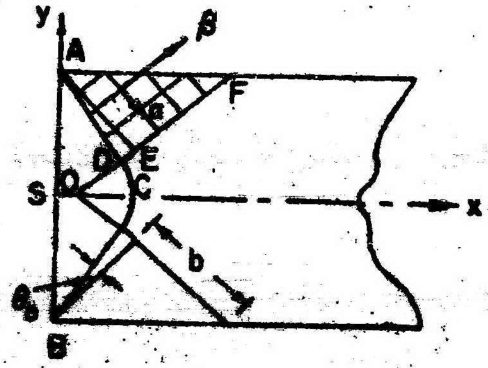

FIG. 2

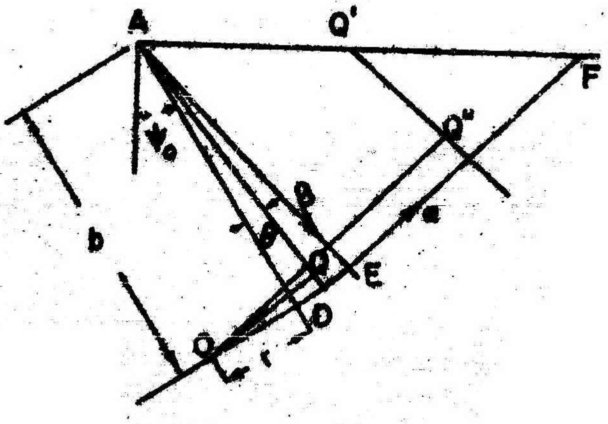

FIG. 3

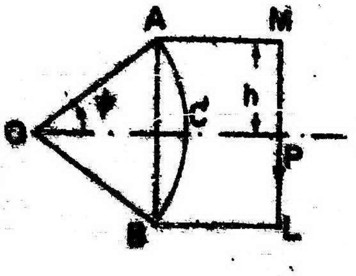
A
M
h
O
C
P
B
L

FIG. 4

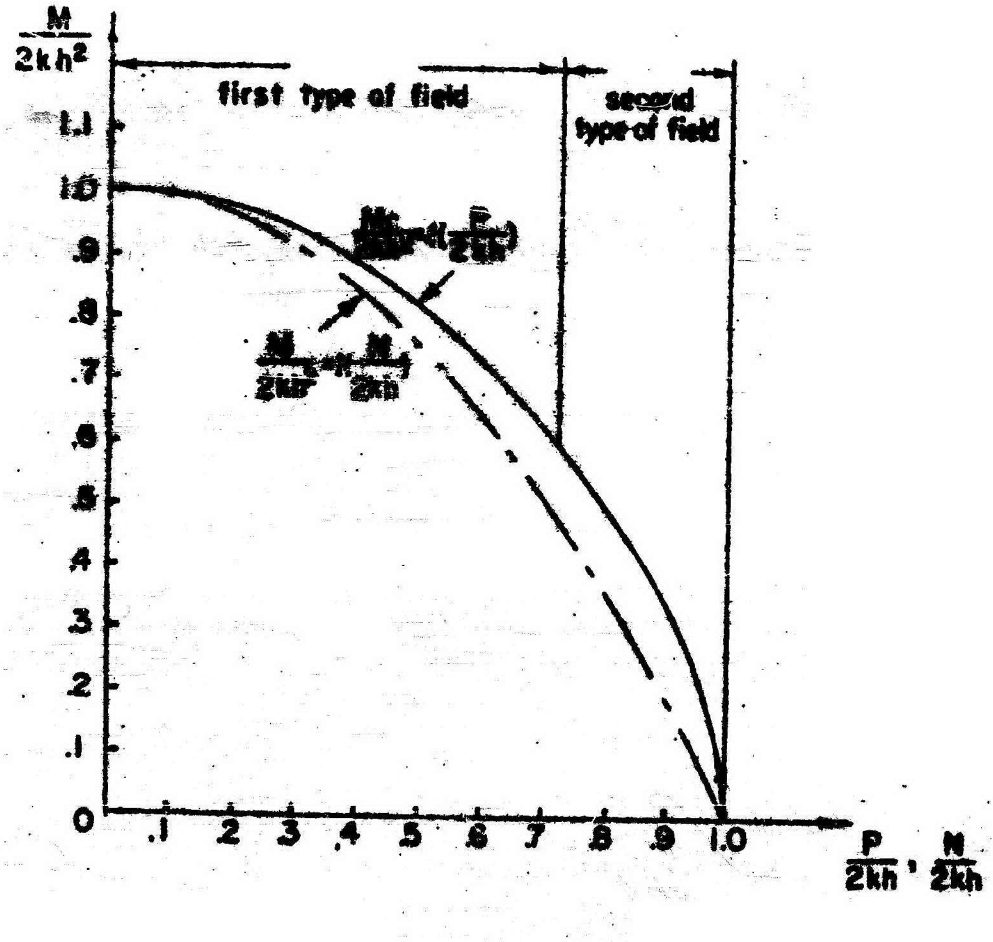
M/2kh²
first type of field
second type of field
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
P/2kh, N/2kh

FIG. 5