EXPERIMENTAL

TOWING TANK

THEORETICAL ASPECTS OF LIMIT CONTROL

by

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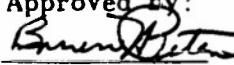
BRUEN H. PETERS

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SUMMARY

This report is concerned with the mathematical features of the design of a control system that has the property of simultaneously reducing the error and the derivatives of the error to zero in a minimum time. The differential equation of the control system is assumed to be of the form

$$Ly = f\left(y, \frac{dy}{dt}, \cdots, \frac{d^{n-1}y}{dt^{n-1}}\right) \quad ,$$

where L is a linear ordinary differential operator of the nth order with constant coefficients, y is the error, and f is the control function or correcting force. In most of the report, it is assumed that the function f is discontinuous and takes on only the values $\overset{+}{-}1$. Thus, the control function acts as a simple switch or relay. The problem is to determine how the function f should be defined, i.e., to determine how to switch the correcting force from $+1$ to -1 and back, so that the error and its derivatives are reduced to zero in a minimum time.

The nth-order equation is reduced to a system of first-order equations and the techniques of matrices are used. The problem is solved for all cases when the eigenvalues of the matrix of the system (or the roots of the characteristic equation of the operator L) are real. The result can be stated as follows: If it is possible to find any control function f that simultaneously reduces the error and its derivatives to zero, then there exists a unique function f which does this in a minimum time. The minimizing function is the one which employs a minimum number of "switches." It should be pointed out that this result holds only for real eigenvalues.

A slightly broader problem is considered in a few simple special cases, namely, the problem where f is restricted only by the inequality $|f| \leq 1$. It is shown in these cases that no reduction in the minimum time is possible, and that the minimizing function is still a function of the discontinuous type, where $f = \overset{+}{-}1$.

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INTRODUCTION

Limit control refers to a discontinuous control that is allowed to take on only extreme values. Such controls are often referred to as bang-bang, relay type, or on-off controls. The present study is concerned with the design of a limit control system that will simultaneously reduce the error and the derivatives of the error to zero in a minimum time.

Mathematically, the error is represented by a differential equation or a system of differential equations where the forcing function or control function is discontinuous and is allowed to take on only extreme values. The problem is to find how to switch the control function from one extreme value to the other in such a way that the error and its derivatives are reduced to zero in a minimum time.

This report extends some of the results presented in Reference 1, which were essentially the first results for this problem. The work is part of the limit control project of the Experimental Towing Tank, Stevens Institute of Technology, under contract with the Office of Naval Research.

STATEMENT OF THE PROBLEM

An Automatic Control Problem

This study is concerned with a minimum problem associated with a linear differential equation containing a discontinuous forcing term. The problem arises physically in the design of an automatic control system for optimum performance.

Consider the control system shown in Figure 1:
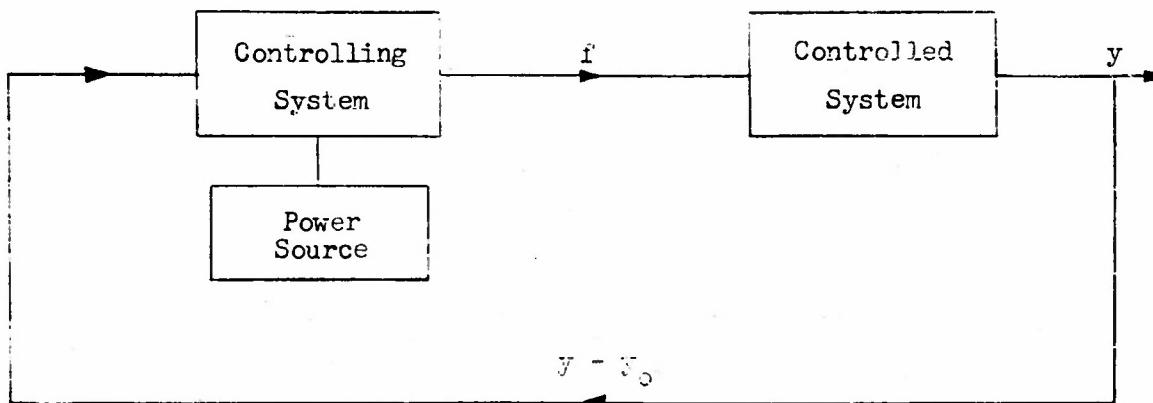
FIGURE 1

The purpose of this system is to keep the output y of the controlled system constant at the value y_c. If, at any instant, the output is different from y_o, then an error-sensitive device feeds the error $y - y_o$ back to the controlling system. The output of the controlling system, i.e., the control function f, is fed into the controlled system and should be designed so as to cause a reduction in the error. The important part of the system is the control function which is supposed to be some function of the error and its time derivatives up to a certain order. The broad problem considered here can now be stated:

To determine the control function so that the error and the derivatives of the error up to a certain order are simultaneously reduced to zero in a minimum time.

In what follows, a drastic restriction is made on the class of control functions that are considered. In fact, in all but one case,

it will be assumed that the control function f is a discontinuous function that is allowed to take on only the values $+b$ and $-b$, where b is a positive constant. The controlling system therefore acts as a switch or a relay that applies the full strength of the power source to the controlled system, either directly, or with a reverse in polarity.

There is no conclusive reason why a relay type control should be the best one for the present problem. However, since the full strength of the power source is always used, it does seem reasonable that the error and its derivatives can be reduced to zero in a shorter time than by any other means. In this connection, it will later be proved, only in the case of very simple systems, that, among all functions satisfying $|f| \leq b$, the minimizing function is of the discontinuous type described.

Mathematical Formulation of the Problem

In the system shown in Figure 1, it is assumed for convenience that the reference value y_0 is zero. The error is then simply $y(t)$. It is also assumed that the controlled system is governed by a linear differential equation with constant coefficients so that the error $y(t)$ is a solution of

$$Ly = f\left(y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}\right), \tag{1}$$

where L represents the operator

$$L \equiv D^n + a_1 D^{n-1} + \ldots a_{n-1}D + a_n \qquad \left(D \equiv \frac{d}{dt}\right), \tag{2}$$

and the control function f is a discontinuous function taking on only the values $+1$ and -1. (If $f = \overset{+}{-}b$, then b can be made to equal 1 by a change in the time scale.) A preliminary statement of the problem is to determine the function f (that is, to determine for what values $f = +1$ and for what values $f = -1$) such that the solution of equation (1) which satisfies the initial conditions

$$y(0) = c_1, \frac{dy(0)}{dt} = c_2, \ldots, \frac{d^{n-1}y(0)}{dt^{n-1}} = c_n \qquad (c_i = \text{constants}),$$

will reach the state

$$y = \frac{dy}{dt} = \ldots = \frac{d^{n-1}y}{dt^{n-1}} = 0$$

in a minimum time.

If the function f is at all complicated, there is some doubt as to what is meant by a solution of equation (1) (suppose, for instance, that f is $+1$ when y is an irrational number and f is -1 otherwise). To circumvent this difficulty, the problem is formulated in different terms. For this purpose, equation (1) is reduced to a system of first-order equations. In the usual manner, let

$$\left. \begin{aligned} y &= x_1 \\ \dot{x}_1 &= x_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= -a_n x_1 - a_{n-1} x_2 - \ldots - a_1 x_n + f \end{aligned} \right\} \tag{3}$$

($\dot{x}_1 = dx_1/dt$ etc.). In matrix form, this can be written as

$$\dot{x} = Ax + p \quad , \tag{4}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} , \quad A = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 \\ \cdot & & & & & & \\ \cdot & & & & & & \cdot \\ 0 & 0 & 0 & . & . & 0 & 1 \\ -a_n & -a_{n-1} & . & . & . & -a_2 & -a_1 \end{bmatrix} , \quad p = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ f(x) \end{bmatrix}$$

Since f can take on only the values ± 1, the solutions of equation (4) should be combinations of the solutions of

$$\dot{x} = Ax \overset{+}{-} e \quad , \tag{5}$$

where $e = \text{col}(0,0,\ldots,0,1)$. The solutions of $\dot{x} = Ax + e$ will be called P-curves (P for positive) and those of $\dot{x} = Ax - e$ will be called N-curves (N for negative). Along each P- or N-curve there is a natural orientation, namely, the direction in which t increases.

A solution of (4) from the point x^o[*] is defined as a continuous curve that consists of a finite sequence of alternating P- and N-arcs (each of finite time length) starting at the point x^o. The solution can be represented by $x(t)$, a continuous vector function of the scalar parameter t. A path from x^o is defined as a solution from x^o which ends at the origin. With each path there is associated a transit time, namely, the sum of the time lengths of the P- and N-arcs making up the path.

With this terminology, the problem can finally be stated as follows:

To find a path of minimum transit time (a minimal path) from each point x^o in the phase space.

Corresponding to each solution of equation (4), as defined above, there is a control function f, namely, $f = +1$ on the P-arcs and $f = -1$ on the N-arcs (here, a P- or an N-arc is regarded as being closed at its initial point and open at its terminal point). The only functions f admitted are those such that all the solutions from a point are of the type described.

Since the problem has been formulated entirely in terms of the system of equations, it is not necessary to restrict the matrix A to be of the above form. The problem can be generalized slightly and instead of considering equation (5), the following equations will be considered:

$$\dot{x} = Ax \overset{+}{-} e \quad , \tag{6}$$

where A is any n by n matrix and e is any non-null vector.[**]

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[*] Superscripts on vectors are indices and not exponents.

[**] It should be remarked that equation (6) is not necessarily equivalent to an n-order equation. However, physical systems arise directly in the form (6), such as the linearized equations that govern the motion of a submerged submarine.

The final formulation of the problem to be treated in this report is:

Consider the system of differential equations

$$\dot{x} = Ax + ef(x) \quad , \tag{7}$$

where A is any n by n matrix, e is any non-null vector, and $f(x)$, a real-valued function of the vector x, can take on only the values ± 1. How should the function $f(x)$ be chosen so that the solution $x(t)$ satisfying arbitrary initial conditions $x(0) = x^0$ reaches the origin $x = 0$ in the least possible time?

General Remarks About the P- and N-Curves.

The existence or nonexistence of minimal paths of course depends on the behavior of the P- and N-curves. Some of the properties of these curves are discussed below.

An important fact is that the family of P-curves is symmetric to the family of N-curves with respect to the origin. This holds for orientation as well as for shape. The symmetry follows from the fact that if $x(t)$ satisfies $\dot{x} = Ax + e$ with $x(0) = x^0$, then $z(t) = -x(t)$ satisfies $\dot{z} = Az - e$ with $z(0) = -x^0$. Using this property, it is necessary to solve the minimal problem for only one-half of the phase space, the solutions for the other half being obtained by symmetry.

It is known from the well-known existence theorem for linear differential equations that there exists a unique P- and N-curve through each point of the phase space. The equations of these curves can be found in terms of the matrix function ϵ^{At}. The solutions of the differential equations

$$\dot{x} = Ax \pm e \tag{8}$$

satisfying the initial conditions $x(0) = x^0$ are

$$x = \epsilon^{At}x^0 \pm \epsilon^{At}\left(\int_0^t \epsilon^{-As} \, ds \right)e \quad , \tag{9}$$

which are the equations of the P- and N-curves. (In the case of an

ambiguous sign, the upper sign holds for the P-curves, the lower sign for the N-curves.)

If the matrix A is nonsingular, (9) can be simplified to

$$x = \epsilon^{At} x^o \overset{+}{-} (\epsilon^{At} - I) A^{-1} e \quad , \tag{10}$$

which can be verified by direct substitution. The points $\overset{+}{-} A^{-1} e$ are singular points (or equilibrium points) of the DE, that is, points where $\dot{x} = 0$. The P-curve through the point $-A^{-1}e$ is just the point itself and similarly for the N-curve through $+A^{-1}e$.

It is well known that the matrix ϵ^{At} can be determined explicitly if the eigenvalues of the matrix A are known. In fact, in order to solve the problem, it will be convenient to make a substitution in the DE that will reduce the matrix A to diagonal or Jordan canonical form where the eigenvalues are explicitly displayed and where the function ϵ^{At} assumes a simple form. Obviously, the character of the P- and N-curves is also determined by the eigenvalues of A. It will be shown, in fact, that the character and existence of minimal paths are determined solely by the nature of the eigenvalues.

In this report, a complete solution to the problem will be given in the case where the eigenvalues of A are all real. The trivial one-dimensional case will be solved first; then the two-dimensional cases will be discussed, and these will be used as a basis for induction in solving the n-dimensional cases. In addition, the broader problem where the control function f is restricted only by the inequality $|f| \leq 1$ will be solved for the cases of the one-dimensional systems and a simple two-dimensional system.

The only known previous results for the present problem are given by Bushaw in Reference 1. Bushaw completely solved the problem for all two-dimensional systems (including the case of complex eigenvalues) that arise from a second order differential equation with constant coefficients.

Before proceeding with the analysis, a few remarks will be made about the general procedure. The problem is to find the minimal paths which, by definition, must enter the origin. The only way a minimal path can reach

the origin is by way of the P- or N-curve through the origin. Let the
curve Γ be defined as the curve obtained by following the P-curve through
the origin backward in time (starting at the origin). The minimal path for
at least some of the points of Γ close to the origin must consist of just
that portion of Γ between the point and the origin (if a minimal path
exists at all). By symmetry, corresponding points are found on the N-curve
through the origin. Thus, all the minimal paths with no "corners" are
found, a "corner" being a junction of nonzero P- and N-arcs. Then the
minimal paths with exactly one corner are determined, and so on, until
all the minimal paths have been found.

ONE- AND TWO-DIMENSIONAL SYSTEMS

One-Dimensional System

The differential equations for a one-dimensional system are simply

$$\dot{x}_1 = ax_1 \overset{+}{-} e_1 \quad , \tag{11}$$

where x_1, a, e_1, are all real numbers. It is assumed that $e_1 > 0$.

Three cases need to be considered, namely $a < 0$, $a > 0$, and $a = 0$ ($\underline{a}$ is actually the eigenvalue). If $a \neq 0$, then the solutions of (11) are

$$x_1 \overset{+}{-} \frac{e_1}{a} = \epsilon^{at}\left(x_1{}^o \overset{+}{-} \frac{e_1}{a}\right) \quad , \tag{12}$$

which are the equations of the P- and N-curves whose initial point is $x_1{}^o$.

Consider first the case $a < 0$. The P-curves start at the point $x_1{}^o$ and continuously approach the point $-e_1/a$ as $t \rightarrow \infty$. The N-curves continuously approach the point $+e_1/a$ as $t \rightarrow \infty$. This behavior is shown schematically in the following Figure:
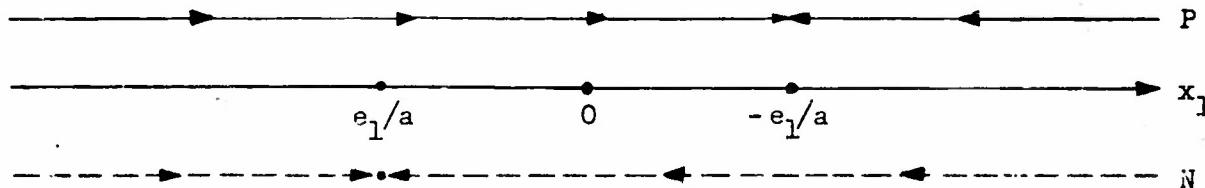
FIGURE 2

Now to find the minimal paths. If $x_1{}^o$ is in the interval $0 < x_1{}^o \leq -e_1/a$, the minimal path is easily seen to be the N-arc from $x_1{}^o$ to the origin. This follows from the fact that the N-arcs in this interval are directed toward the left while the P-arcs are directed toward the right. If the minimal path contained a P-arc, this would mean that x_1 would be increased and, in order to reach the origin, a longer N-arc would have to be traversed.

If $x_1{}^o$ is in the interval $x_1{}^o > -e_1/a$, both the N-arcs and the

P-arcs are directed toward the left. However, the minimal path is still the N-arc connecting the point $x_1^{\,0}$ with the origin. Because of the result in the preceding paragraph, it is only necessary to show that the minimal path cannot contain a P-arc in the interval $x_1 > -e_1/a$. If the minimal path did contain a P-arc, say between the points $x_1^{\,0}$ and x_1, where $x_1^{\,0}$ and x_1 are greater than $-e_1/a$ and $x_1 < x_1^{\,0}$, the time to traverse this arc could be easily obtained from (12) to be

$$t_P = -\frac{1}{a}\ln\left(\frac{x_1^{\,0} + e_1/a}{x_1 + e_1/a}\right) . \tag{13}$$

The time to traverse an N-arc connecting the same points is

$$t_N = -\frac{1}{a}\ln\left(\frac{x_1^{\,0} - e_1/a}{x_1 - e_1/a}\right) . \tag{14}$$

It is easy to prove that $t_N < t_P$ and therefore that the minimal path cannot contain a P-arc.

By symmetry, an analagous result is obtained for the points $x_1^{\,0} < 0$ where N-arcs are replaced by P-arcs. Hence, a unique minimal path exists from every initial point.

If the eigenvalue $\underline{a}$ is positive, the equations for the P- and N-curves are still given by (12) but the directions are reversed, as shown in the following Figure:
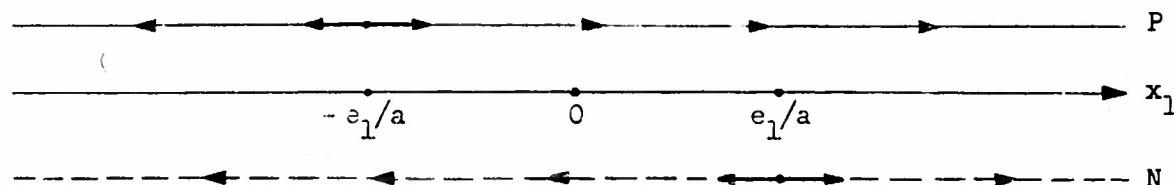

FIGURE 3

It is seen immediately that no paths exist if $\left|x_1^{\,0}\right| > e_1/a$. If $\left|x_1^{\,0}\right| \le e_1/a$, the same procedure as above proves that a unique minimal

path exists. If $0 < x_1^o \leq e_1/a$, the minimal path is the N-arc connecting x_1^o with the origin, and if $-e_1/a \leq x_1^o < 0$, the minimal path is the P-arc connecting x_1^o with the origin.

If $a = 0$, the solution (12) fails. The equations of the P- and N-curves in this case are

$$x_1 = \pm e_1 t + x_1^o \quad . \tag{15}$$

Here, the P-curve from any point is directed to the right and the N-curve is directed to the left. By using the same reasoning as above, it is seen that a unique minimal path exists from every point x_1^o; if $x_1^o > 0$, the minimal path is the N-arc through x_1^o , while if $x_1^o < 0$, the minimal path is the P-arc through x_1^o .

The results of this section are summarized in the following theorem:

Theorem 1: In the one-dimensional system

$$\dot{x}_1 = ax_1 + e_1 f(x_1) \quad ,$$

where $e_1 > 0$ and $f(x_1)$ is restricted to the values ± 1 , a unique minimal path exists if $a \leq 0$; if $a > 0$, a unique minimal path exists for points x_1^o such that $|x_1^o| \leq e_1/a$ and no path exists for $|x_1^o| > e_1/a$. The minimal path (if it exists) for $x_1^o > 0$ is just the N-arc connecting x_1^o with the origin, and for $x_1^o < 0$, the P-arc connecting x_1^o with the origin. (In terms of the control function f , this means $f(x_1) = -\text{signum } x_1$.)

One-Dimensional System with $|f(x_1)| \leq 1$

The broader problem will be considered here for the one-dimensional system

$$\dot{x}_1 = ax_1 + e_1 f(x_1) \quad , \tag{16}$$

where $e_1 > 0$ and $|f(x_1)| \leq 1$. In order to insure solutions of the differential equation, it will be assumed that $f(x_1)$ is piecewise

continuous with only a finite number of discontinuities in any bounded interval. The solution of (16) through any initial point will then be uniquely determined by the sole requirement of continuity. The class of functions considered here will be broader than the class of functions considered in the preceding section. It will be shown, however, that the minimum time obtained in the last section cannot be improved.

Theorem 2: Among all functions $f(x_1)$ satisfying the conditions in the preceding paragraph, the minimizing function is just the function $f(x_1) = -\text{signum } x_1$ that was obtained in Theorem 1.

This theorem will be proved only for the case $a < 0$. The other cases can be treated in a similar manner. If the initial point is $x_1^o > 0$, the time τ for $f(x_1) = -\text{signum } x_1$ to get to the origin is simply

$$\tau = -\frac{1}{a} \ln \left(\frac{x_1^o - e_1/a}{-e_1/a} \right) \quad . \tag{17}$$

It will be shown that the time for any other f will exceed τ. Since $|f| \leq 1$ and $e_1 > 0$,

$$-e_1 \leq e_1 f(x_1) \leq e_1 \tag{18}$$

or, using the differential equation (16),

$$-e_1 \leq \dot{x}_1 - ax_1 \leq e_1 \quad . \tag{19}$$

Multiplying (19) by the integrating factor ϵ^{-at} gives

$$-e_1 \epsilon^{-at} \leq \frac{d}{dt} (x_1 \epsilon^{-at}) \leq e_1 \epsilon^{-at} \quad . \tag{20}$$

Integrating this inequality from 0 to t, remembering that $x_1(0) = x_1^o$, gives

$$\frac{e_1}{a} \epsilon^{-at} - \frac{e_1}{a} \leq x_1 \epsilon^{-at} - x_1^o \leq -\frac{e_1}{a} \epsilon^{-at} + \frac{e_1}{a} \quad . \tag{21}$$

Setting $x_1 = 0$ and solving the left-hand inequality results in the following inequality for the time to reach the origin:

$$t \geq -\frac{1}{a} \ln\left(\frac{x_1^0 - e_1/a}{-e_1/a}\right) \quad .$$

(22)

This shows that $t \geq \tau$. It is easy to see that the strict inequality will hold unless the function $f(x_1)$ is identical with the minimizing function $f(x_1) = -\text{signum } x_1$. This completes the proof.

Two-Dimensional Systems

The minimal problem will be considered for a two-dimensional system

$$\dot{x} = Ax \pm e \qquad (e \neq 0) \quad , \qquad (23)$$

where A is a 2 by 2 matrix and x and e are two-dimensional vectors. Only those cases where A has <u>real eigenvalues</u> will be considered.

It is convenient to make a substitution in the DE, namely,

$$x = Tz \quad , \qquad (24)$$

where T is a real nonsingular matrix. The DE becomes

$$\dot{z} = Dz \pm w \quad , \qquad (25)$$

where $D = T^{-1}AT$ and $w = T^{-1}e$. The matrix T can be chosen so that D assumes one of the three following forms:[*]

$$\left. \begin{array}{l}
\text{Case (a):} \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda \end{bmatrix} \quad , \\[2em]
\text{Case (b):} \quad D = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \quad , \\[2em]
\text{Case (c):} \quad D = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad ,
\end{array} \right\} \qquad (26)$$

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[*] See, e.g., Reference 2, page 208.

where λ_1, λ are the eigenvalues of A. Case (a) will be obtained if the eigenvalues are distinct, case (b), if the eigenvalues are equal but there are two linearly independent eigenvectors, and case (c), if the eigenvalues are equal with only one independent eigenvector.

The solutions of (25) are

$$z = \epsilon^{Dt} z^0 \pm \epsilon^{Dt} \left\{ \int_0^t \epsilon^{-Ds} \, ds \right\} w \quad , \tag{27}$$

where, corresponding to each of the forms of D in (26),

$$\text{Case (a):} \quad \epsilon^{Dt} = \begin{bmatrix} \epsilon^{\lambda_1 t} & 0 \\ 0 & \epsilon^{\lambda t} \end{bmatrix} \quad , \left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\right\}$$

$$\text{Case (b):} \quad \epsilon^{Dt} = \begin{bmatrix} \epsilon^{\lambda t} & 0 \\ 0 & \epsilon^{\lambda t} \end{bmatrix} \quad , \tag{28}$$

$$\text{Case (c):} \quad \epsilon^{Dt} = \begin{bmatrix} \epsilon^{\lambda t} & t\epsilon^{\lambda t} \\ 0 & \epsilon^{\lambda t} \end{bmatrix} \quad .$$

It will be necessary to consider several cases in order to solve the minimal problem. However, there are some results that can be established for all two-dimensional systems. In this connection, let Γ^+ be the curve obtained by starting at the origin and following the P-curve backwards in time. Γ^+ is the curve

$$z = \epsilon^{Dt} \left\{ \int_0^t \epsilon^{-Ds} \, ds \right\} w \qquad (t < 0) \quad . \tag{29}$$

Also, let Γ^- be the N-curve symmetric to Γ^+ with respect to the origin. The following theorem will now be proved:

Theorem 3: In the two-dimensional system

$$\dot{x} = Ax \pm e \qquad (e \neq 0) \quad ,$$

where A has real eigenvalues, the minimal paths for points on $\Gamma^+ + \Gamma^-$ are just the portions of Γ^+ or Γ^- connecting the initial point and the origin.

Let z^0 be any point on Γ^+, except the origin. Let the minimal path described in Theorem 3 be denoted by $\Gamma^+_{z^0}$ and let S be any other path from z^0. It must be shown that the time length of $\Gamma^+_{z^0}$ is less than the time length of S. This will be done by projecting both paths on the z_2-axis. It will turn out that the time lengths of the paths are not altered by projection. It will further turn out that the projected paths are paths in the z_2-axis for the one-dimensional minimal problem which was considered earlier. From (27), the equation of the z_2-component of any P- or N-curve can be obtained, the result being the same regardless of which of the three forms the matrix D takes. This is easily seen to be

$$z_2 \overset{+}{-} \frac{w_2}{\lambda} = \epsilon^{\lambda t}\left(z^0_2 \overset{+}{-} \frac{w_2}{\lambda}\right) \tag{30}$$

if $\lambda \neq 0$ and

$$z_2 = \overset{+}{-}w_2 t + z^0_2 \tag{31}$$

if $\lambda = 0$. It can be seen that the time length of a P- or an N-arc depends only on the z_2-coordinates and is not altered by projection on the z_2-axis. It is also seen that (30) and (31) are just the P- and N-curves of the one-dimensional problem

$$\dot{z}_2 = \lambda z_2 \overset{+}{-} w_2 \tag{32}$$

which was discussed earlier. For the one-dimensional problem, the projection of $\Gamma^+_{z^0}$ is the P-arc connecting z^0_2 with the origin, which is just the minimal path for the one-dimensional problem. The projection of S, which must contain at least one N-arc, will be different from the unique minimal path of the one-dimensional problem. The time length of $\Gamma^+_{z^0}$ is therefore less than the time length of S. This completes the proof.

In some degenerate cases, the minimal paths just found are the only ones. These cases are given by the following theorem:

Theorem 4: If e is an eigenvector of the matrix A, then minimal paths exist only for points on Γ^+ or Γ^-.

It will be shown that the P-curve through the origin is a straight line and therefore so is the N-curve through the origin (by symmetry). Since the N-curve and the P-curve coincide and since the N- or P-curves through any point are unique, no other Γ- or N-curve can intersect Γ^+ or Γ^-. Thus, there is no way to reach the origin from a point not on Γ^+ or Γ^-.

In terms of the z-coordinates, the equation of the P-curve through the origin is

$$z = \epsilon^{Dt} \int_0^t \epsilon^{-Ds} \, ds \cdot w \quad . \tag{33}$$

Differentiating (33) gives

$$\dot{z} = \epsilon^{Dt} \int_0^t \epsilon^{-Ds} \, ds \cdot Dw + w \quad . \tag{34}$$

Clearly, e is an eigenvector of A if and only if w is an eigenvector of D. Therefore, $Dw = \lambda w$ and (34) becomes

$$\dot{z} = \lambda \epsilon^{Dt} \int_0^t \epsilon^{-Ds} \, ds \cdot w + w \tag{35}$$

or

$$\dot{z} = \lambda z + w \quad .$$

This is the equation of a straight line and the proof is complete.

The eigenvectors of D, corresponding to cases (a), (b), (c) of (26), are

Case (a):
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

Case (b): any vector ,

$$\left.\begin{array}{c} \end{array}\right\} \quad (36)$$

Case (c): $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

According to the theorem just proved, case (b) is a degenerate case and need not be considered further. To avoid degenerate cases in (a) and (c), and for convenience, it is assumed in the sequel that

$$w_i > 0 \qquad (i = 1,2) \quad . \qquad (37)$$

Distinct Nonzero Eigenvalues

In this case, the matrix D assumes a diagonal form. The DE's are in the uncoupled form

$$\dot{z} = Dz \overset{+}{-} w \quad , \qquad (38)$$

where

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} , \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} , \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad .$$

λ_1 and λ_2 are the eigenvalues and $\lambda_1 \neq \lambda_2$. The equations of the P- and N-curves are

$$z \overset{+}{-} u = \epsilon^{Dt}(z^o \overset{+}{-} u) \quad , \qquad (39)$$

where

$$u = D^{-1}w = \begin{bmatrix} \dfrac{w_1}{\lambda_1} \\ \dfrac{w_2}{\lambda_2} \end{bmatrix} , \quad \epsilon^{Dt} = \begin{bmatrix} \epsilon^{\lambda_1 t} & 0 \\ 0 & \epsilon^{\lambda_2 t} \end{bmatrix} \quad .$$

The nature of the P- and N-curves depends on the sign of λ_1 and λ_2. There are three possibilities: (1) $\lambda_1 < 0$, $\lambda_2 < 0$, (2) $\lambda_1 > 0$, $\lambda_2 > 0$, and (3) $\lambda_1 < 0$, $\lambda_2 > 0$. These correspond to the cases where the singularity of the differential equation is (1) a stable node, (2) an unstable node, and (3) a saddle point. Sketches of the P- and N-curves are shown in Figures 4, 5, and 6, pages 20 and 21. These Figures are drawn under the assumptions that $w_1 > 0$ and $w_2 > 0$. In case (1), the P- and N-curves are semi-parabola-like curves that tend to $\mp u$ as $t \to +\infty$. In case (2), the curves have the same shape but with reverse direction. The reason for the interchange of $\overset{+}{-}u$ is that the components of w are assumed to be positive. In (3), the P- and N-curves are hyperbola-like curves around the points $\mp u$.

The case of distinct negative eigenvalues will be discussed first. This will serve as a model for the rest of the proofs. The minimal paths from points of Γ^+ and Γ^- are already known. It is necessary to find minimal paths from all the other points in the plane.

Let C be the simple curve $\Gamma^+ + \Gamma^- + 0$. This curve divides the plane into two open regions. The region which contains the point $+u$

FIGURE 7

$$\lambda_1 < 0 \; , \; \lambda_2 < 0$$

FIGURE 4

P-curves
N-curves
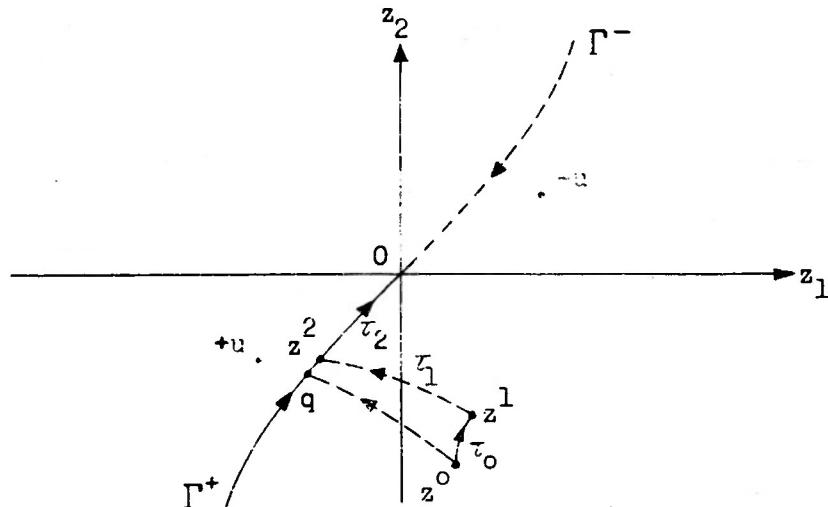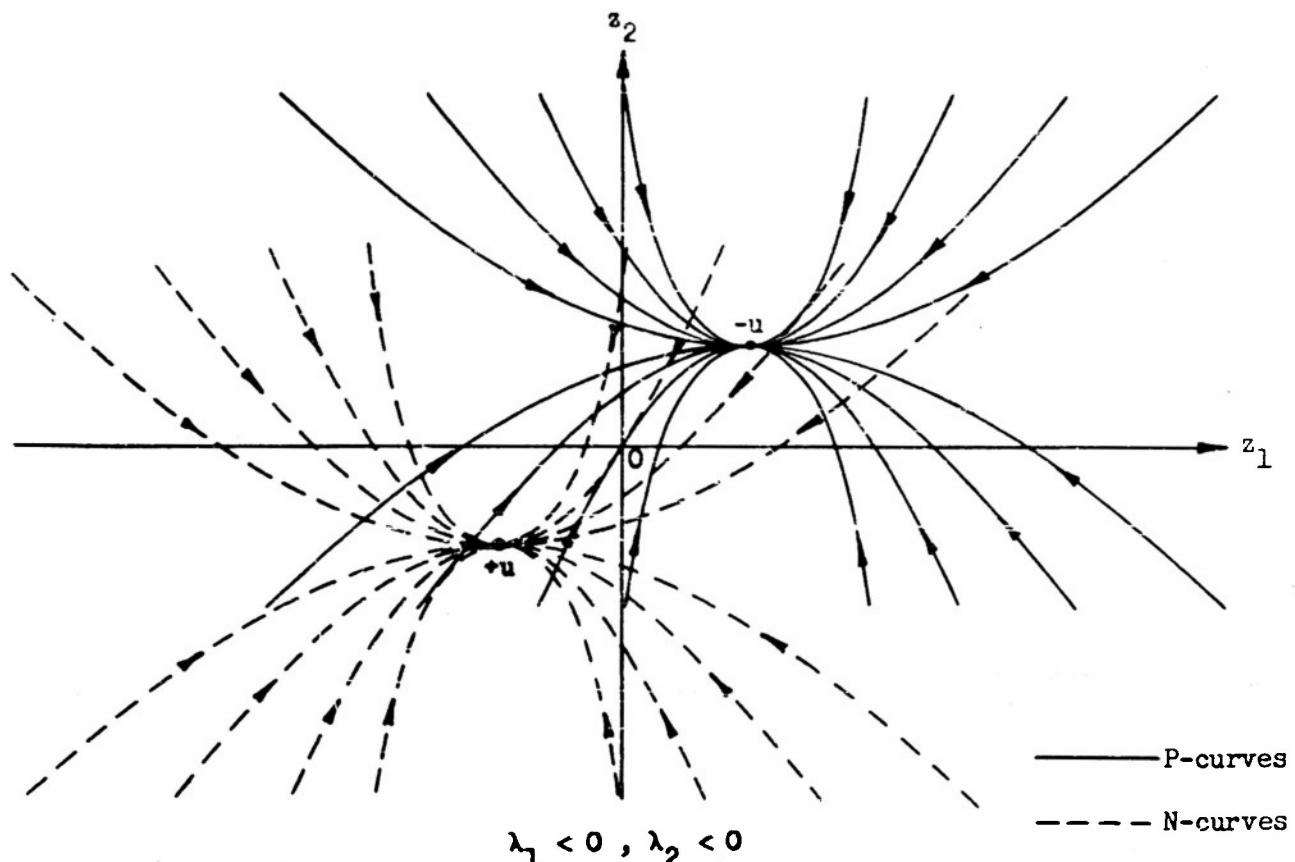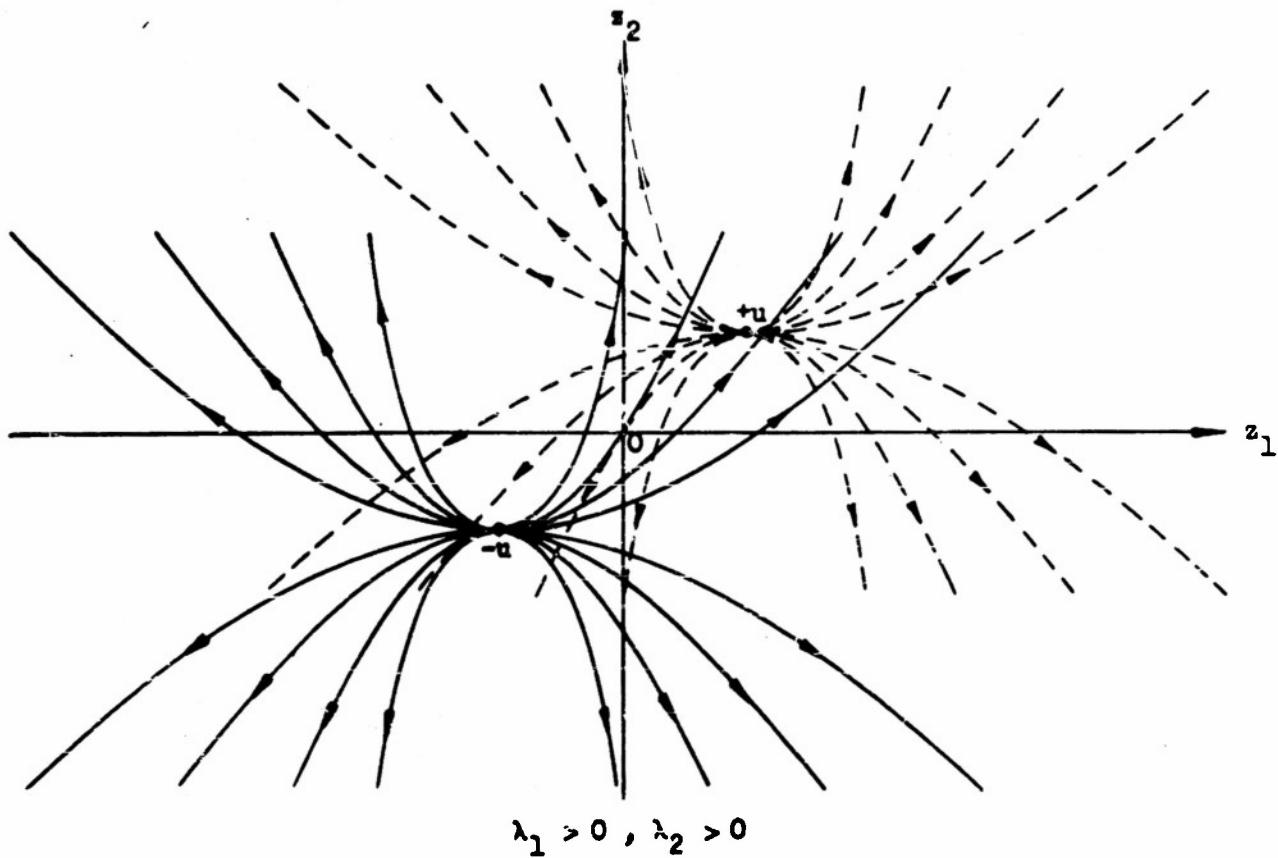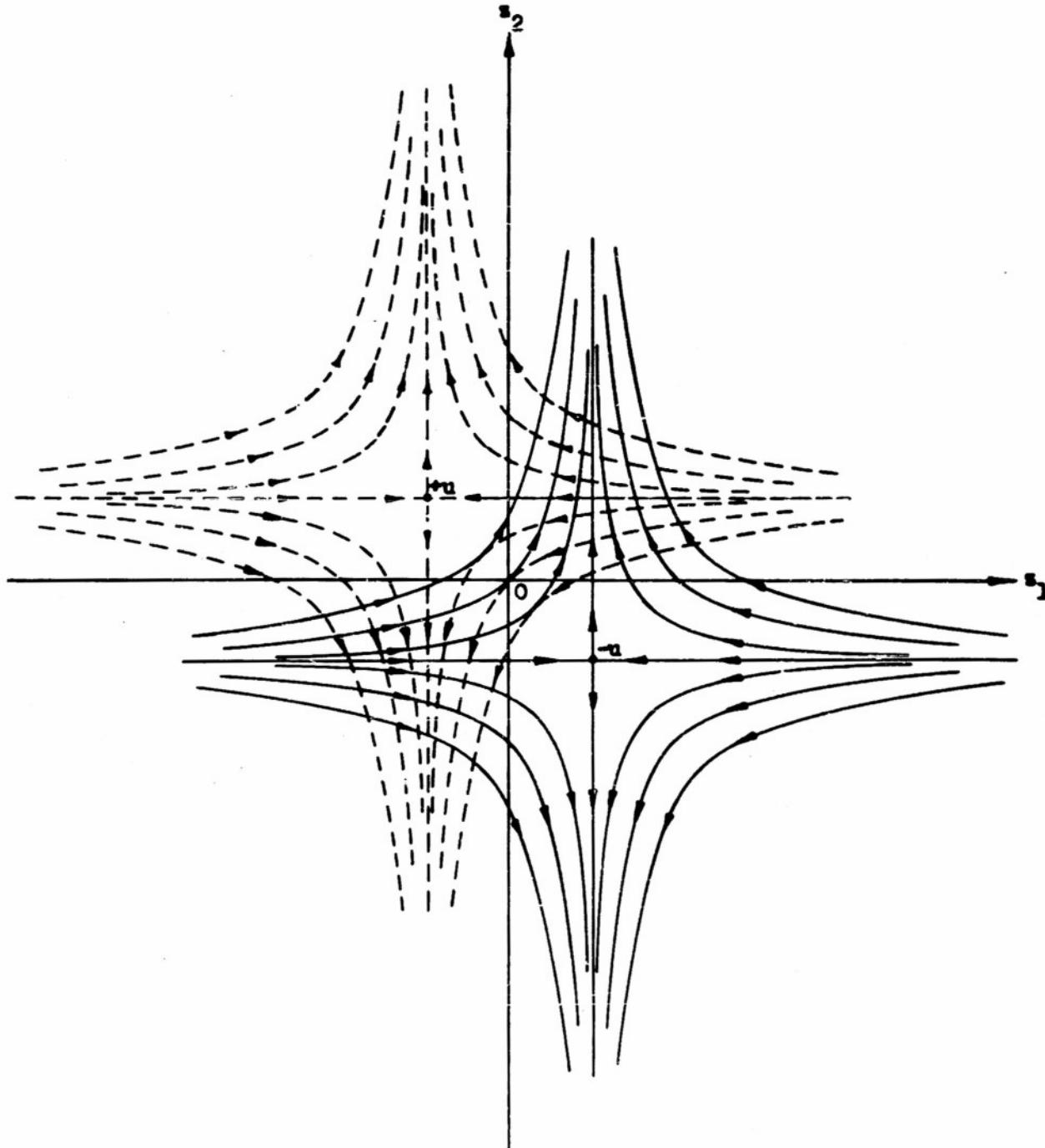
$$\lambda_1 > 0 \; , \; \lambda_2 > 0$$

FIGURE 5

$$\lambda_1 < 0 \; , \; \lambda_2 > 0$$

FIGURE 6

will be called R^+, and the region which contains the point $-u$ will be called R^-. If z^o is any point in R^-, the N-arc from z^o must intersect Γ^+ at some point q, since the N-arc connects z^o and $+u$ which are on opposite sides of C. Also, this N-arc cannot intersect Γ^- since Γ^- is an N-arc and the N-arc through any point is unique. Thus, $z^o q0$ is a path from z^o (it will turn out that this is the minimal path). A similar argument applied to points of R^+ shows that at least one path exists from every point in the plane. The points in R^- can also be obtained by starting at points of Γ^+ and following the N-arc through the point backwards in time, that is, for a time $t < 0$. A similar statement holds for points in R^+.

The following theorem provides the minimal paths:

Theorem 5: In the case of distinct negative eigenvalues, a unique minimal path exists from every point in the plane. If z^o is in R^-, the minimal path consists of following the N-arc through z^o until it intersects Γ^+ and then following Γ^+ to the origin. The minimal paths for points in R^+ are obtained by symmetry. (In terms of the control function f, this means f is -1 in R^- and on Γ^-, and f is $+1$ in R^+ and on Γ^+.)

Let z^o be any point in R^-. If any path from z^o leaves R^-, it must pass through the curve Γ^+. The path cannot leave R^- by passing through Γ^- since, as can easily be proved, all curves, necessarily P-curves, intersecting Γ^- point into R^-. After a path intersects Γ^+, the minimal path from that point must coincide with Γ^+. Therefore, only those paths from z^o that lie entirely in R^- and whose last corner is on Γ^+ need be considered.

Let the minimal path described in Theorem 5 be called Δ. It is known that Δ exists and is unique; also, Δ has exactly one corner. Δ is the path $z^o q0$ in Figure 7. It will be shown that the time length of Δ is less than any path that contains two corners. Successive use of this result will prove that Δ has shorter length than any other path.

Consider any path from z^o with two corners, that is, for a time τ_o, follow a P-arc from z^o to a point z^1, then, for a time τ_1, follow

an N-arc to a point z^2 (on Γ^+), then for a time τ_2, follow Γ^+ to the origin. If the point z^0 is kept fixed, τ_0 can be taken as the independent variable, since given τ_0, then τ_1 and τ_2 can be determined uniquely. Let the total time length of this path be τ, that is,

$$\tau = \tau_0 + \tau_1 + \tau_2 \qquad (\tau, \tau_0, \tau_1, \tau_2 > 0) \quad . \qquad (40)$$

It will be shown that

$$\frac{d\tau}{d\tau_0} > 0 \quad \text{for} \quad \tau_0 > 0 \quad . \qquad (41)$$

The minimum time will then occur when $\tau_0 = 0$. However, $\tau_0 = 0$ gives the desired path Δ.

By using the equations of the P- and N-curves, the following equations for z^0, z^1, and z^2 are obtained:

$$\left.\begin{array}{l} z^2 + u = \epsilon^{-D\tau_2} u \\[2mm] z^1 - u = \epsilon^{-D\tau_1} (z^2 - u) \\[2mm] z^0 + u = \epsilon^{-D\tau_0} (z^1 + u) \quad . \end{array}\right\} \qquad (42)$$

By successive elimination, it is found that

$$z^0 + u = \left\{ \epsilon^{-D(\tau_0 + \tau_1 + \tau_2)} - 2\epsilon^{-D(\tau_0 + \tau_1)} + 2\epsilon^{-D\tau_0} \right\} u \qquad (43)$$

or, taking components,

$$z^0_i + u_i = \left\{ \epsilon^{-\lambda_i \tau} - 2\epsilon^{-\lambda_i(\tau_0 + \tau_1)} + 2\epsilon^{-\lambda_i \tau_0} \right\} u_i \quad (i = 1,2) \quad . \qquad (44)$$

Differentiating both sides with respect to τ_0 and solving for $d\tau/d\tau_0 = \dot{\tau}$ gives, after some simplification,

$$\dot{\tau} - 2\epsilon^{\lambda_i \tau_2} (\dot{\tau}_1 + 1) = -2\epsilon^{\lambda_i(\tau_1 + \tau_2)} \qquad (i = 1,2) \quad . \qquad (45)$$

These are two simultaneous equations for $\dot{\tau}$ and $(\dot{\tau} + 1)$. Solving for $\dot{\tau}$ by determinants results in

$$\dot{\tau} = 2\epsilon^{(\lambda_1 + \lambda_2)\tau_2} \frac{\begin{vmatrix} 1 & \epsilon^{\lambda_1 \tau_1} \\ 1 & \epsilon^{\lambda_2 \tau_1} \end{vmatrix}}{\begin{vmatrix} 1 & \epsilon^{\lambda_1 \tau_2} \\ 1 & \epsilon^{\lambda_2 \tau_2} \end{vmatrix}} \qquad (46)$$

The determinant in the denominator is not zero since $\lambda_1 \neq \lambda_2$ and $\tau_2 > 0$. Clearly, the quotient of the determinants will be positive since the numerator will have the same sign as the denominator. It has thus been shown that $\dot{\tau} > 0$ for $\tau_0 > 0$, and the proof is complete.

The case of positive distinct eigenvalues can now be disposed of fairly readily. A study of Figure 5 suggests that no paths exist from points outside the open region Q bounded by the P-curve through $+u$ (followed backwards) and the N-curve through $-u$ (followed backwards) as shown below:
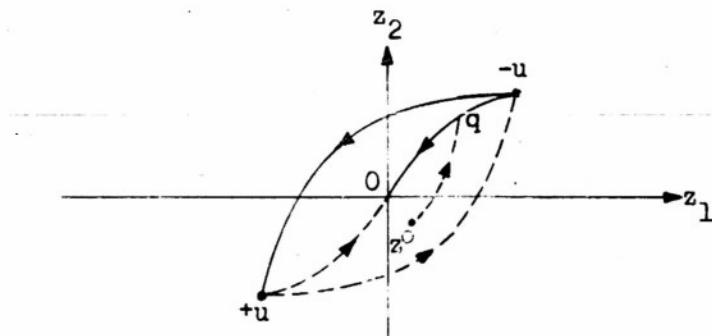

FIGURE 8

Of course this can be proved analytically (see Reference 1) and is based on the fact that all curves (necessarily N-curves) intersecting the P-arc connecting $-u$ and $+u$ are directed out of the region Q.

The proof given for Theorem 5 holds good without change for points in Q since the only place that the sign of the eigenvalues came into the proof was in guaranteeing the existence of paths. The region R^- is now the region in Q and between the curve $C = \Gamma^+ + \Gamma^- + 0$ and the N-curve through $-u$, and R^+ is the reflection of R^- in the origin. The following theorem therefore holds (Figure 8 shows a typical minimal path):

Theorem 6: In the case of positive distinct eigenvalues, a minimal path exists (and is unique) only in the region Q defined above. If the regions R^+ and R^- are redefined as above, the minimal paths are the same as in Theorem 5.

In the case where $\lambda_1 < 0$ and $\lambda_2 > 0$, the only changes necessary for the above theorem to remain true are in the definitions of Q, R^+, R^-. Q is easily determined to be all points in the horizontal strip $-u_2 < z_2 < u_2$. The region R^- is the region between the curve C and the line $z_2 = u_2$, and R^+, its reflection in the origin. A typical minimal path is shown in Figure 9:
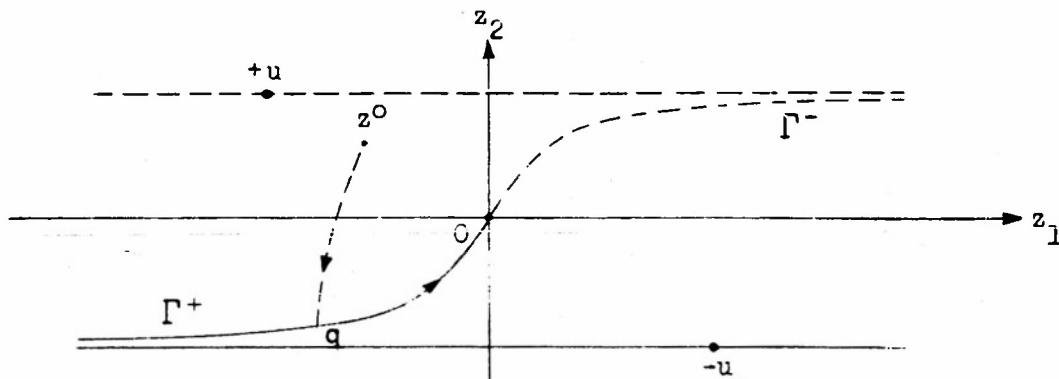

FIGURE 9

Distinct Eigenvalues with One Eigenvalue Zero

The matrix D still assumes the diagonal form

$$D = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \qquad (\lambda \neq 0) \qquad . \tag{47}$$

The solutions of the DE $\dot{z} = Dz \overset{+}{-} w$ are different from those of the last section since D is now a singular matrix. The solutions are

$$z_1 \overset{+}{-} \frac{w_1}{\lambda} = \epsilon^{\lambda t}\left(z^0_1 \overset{+}{-} \frac{w_1}{\lambda}\right)$$

$$z_2 = z^0_2 \overset{+}{-} w_2 t \quad . \tag{48}$$

The sketches of these P- and N-curves are shown in Figures 10 and 11, page 27, for the cases $\lambda < 0$ and $\lambda > 0$, respectively.

It can be seen that if $\lambda < 0$, a path exists from every point in the plane while if $\lambda > 0$, paths exist only from the points $|z_1| < |w_1/\lambda|$. The same method of proof will be used as in the last section. The theorem to be proved will not be reformulated since it is the same as Theorem 6, where Q is the region from which paths exist as defined above, and R^- and R^+ are the regions above and below the curve C, respectively.

The notation of Theorem 5 will be used. It is only necessary to find the equation for $\dot{\tau}$ based on the new P- and N-curves and see if $\dot{\tau} > 0$. Since the first equation of (48) is the same as the z_1-component of (39), one equation for $\dot{\tau}$ is (45):

$$\dot{\tau} - 2\epsilon^{\lambda \tau_2}(\dot{\tau}_1 + 1) = -2\epsilon^{\lambda(\tau_1 + \tau_2)} \quad . \tag{49}$$

The second equation can easily be found by using the second equation of (48). The equations similar to (42) are

$$z^2_2 = w_2(-\tau_2)$$

$$z^1_2 = z^2_2 - w_2(-\tau_1)$$

$$z^0_2 = z^1_2 + w_2(-\tau_0) \quad . \tag{50}$$

By elimination, it is found that

$\lambda < 0$
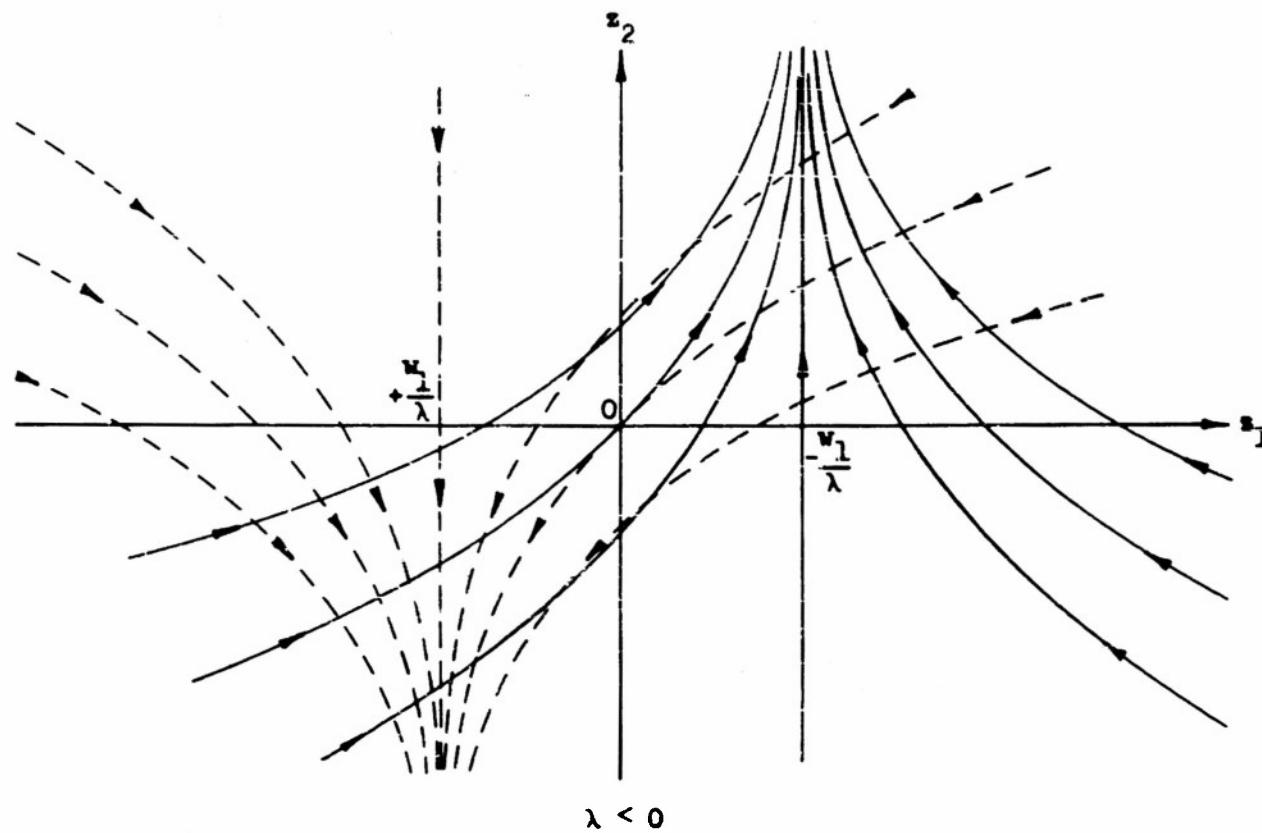
FIGURE 10

$\lambda > 0$
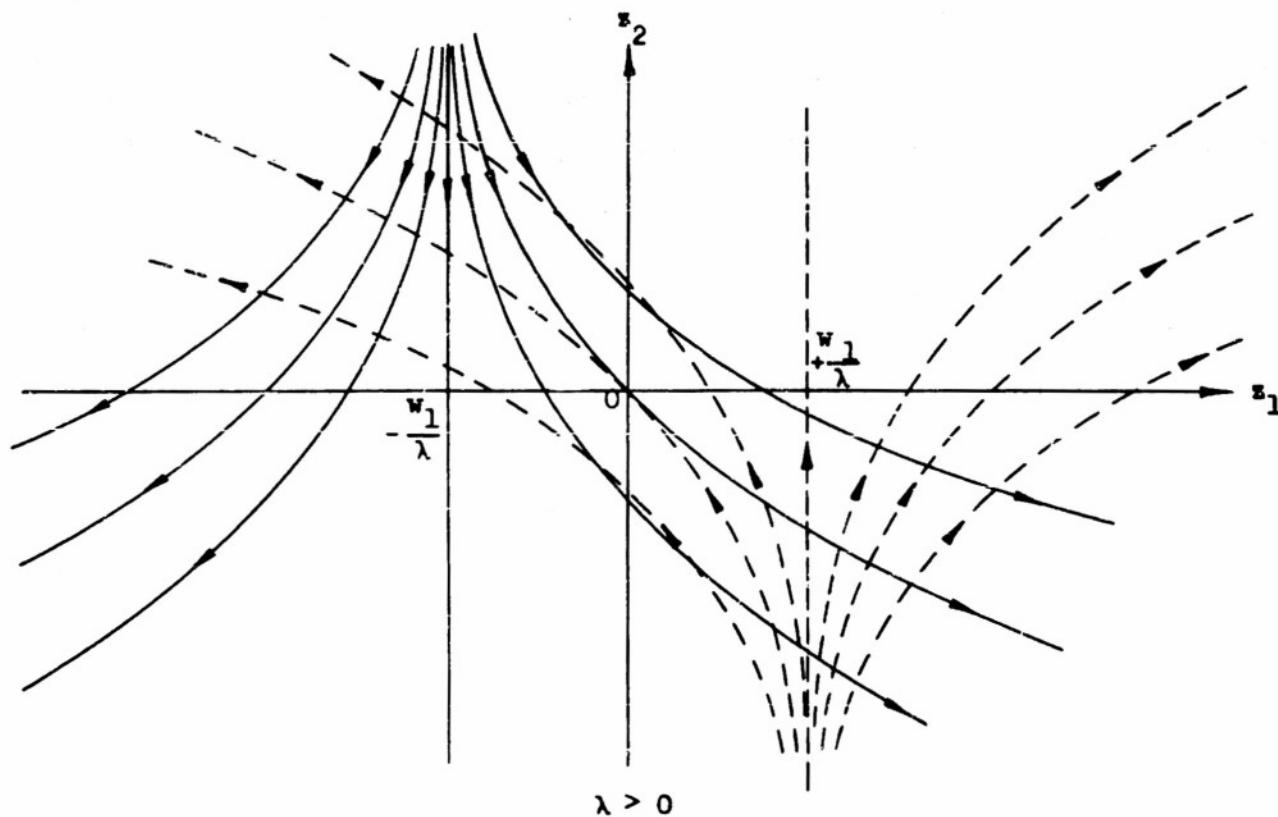
FIGURE 11

$$z^o_2 = \ddot{w}_2(-\tau_2 + \tau_1 - \tau_o) \quad , \tag{51}$$

which is differentiated with respect to τ_o to obtain

$$\dot{\tau}_1 - \dot{\tau}_2 - 1 = 0 \quad . \tag{52}$$

It is recalled that $\tau = \tau_o + \tau_1 + \tau_2$; differentiating this equation and combining it with (52) gives

$$\dot{\tau}_1 = \frac{\dot{\tau}}{2} \quad . \tag{53}$$

Substituting this into (49) and solving for $\dot{\tau}$ results in

$$\dot{\tau} = 2\epsilon^{\lambda\tau_2} \frac{1 - \epsilon^{\lambda\tau_1}}{1 - \epsilon^{\lambda\tau_2}} \quad . \tag{54}$$

Since $\lambda \neq 0$ and $\tau_2, \tau_1 > 0$, it is seen that $\dot{\tau} > 0$ for $\tau_o > 0$, which completes the proof.

Repeated Eigenvalues

The matrix D now takes the form

$$D = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad . \tag{55}$$

If $\lambda \neq 0$, the P- and N-curves are

$$z \overset{+}{-} u = \epsilon^{Dt}(z^o \overset{+}{-} u) \quad , \tag{56}$$

where

$$u = D^{-1}w \quad \text{and} \quad \epsilon^{Dt} = \epsilon^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad ;$$

or if $\lambda = 0$, the equations are

$$z_1 = \pm w_2 \frac{t^2}{2} + t(z^0_2 \pm w_1) + z^0_1$$

$$z_2 = \pm w_2 t + z^0_2 \quad . \tag{57}$$

The sketches of these curves are shown in Figures 12, 13, 14, pages 30 and 31. It can be seen that, if $\lambda \leq 0$, paths exist from every point in the plane, while if $\lambda > 0$, paths exist only in a region Q bounded by the P-curve through $+u$ followed backwards and the N-curve through $-u$ followed backwards.

The region R^- is the region above the curve C if $\lambda \leq 0$ and the region below C if $\lambda > 0$. With this notation, Theorem 6 remains true. The proof is essentially the same as that of Theorem 6 and will not be given. The only change is in the algebra due to the different equations for the P- and N-curves.

Summary of Two-Dimensional Cases

Some definitions are reviewed below for convenience in summarizing the results of the two-dimensional cases:

Let Γ^+ be the point set obtained by following the P-curve through the origin for all time $t < 0$.

Let R^- be the point set obtained by starting at each point of Γ^+ and following the N-curve through the point for all time $t < 0$.

Let Γ^- and R^+ be the point sets obtained by reflecting Γ^+ and R^- in the origin.

Let $C = \Gamma^+ + \Gamma^- + 0$.

Let $Q = R^+ + R^- + C$.

The following theorem summarizes the results:

Theorem 7: In the two-dimensional system

$$\dot{x} = Ax + ef(x) \quad ,$$

where $e \neq 0$, $f = \pm 1$, and A has real eigenvalues, a unique minimal

$\lambda < 0$

FIGURE 12

$\lambda > 0$
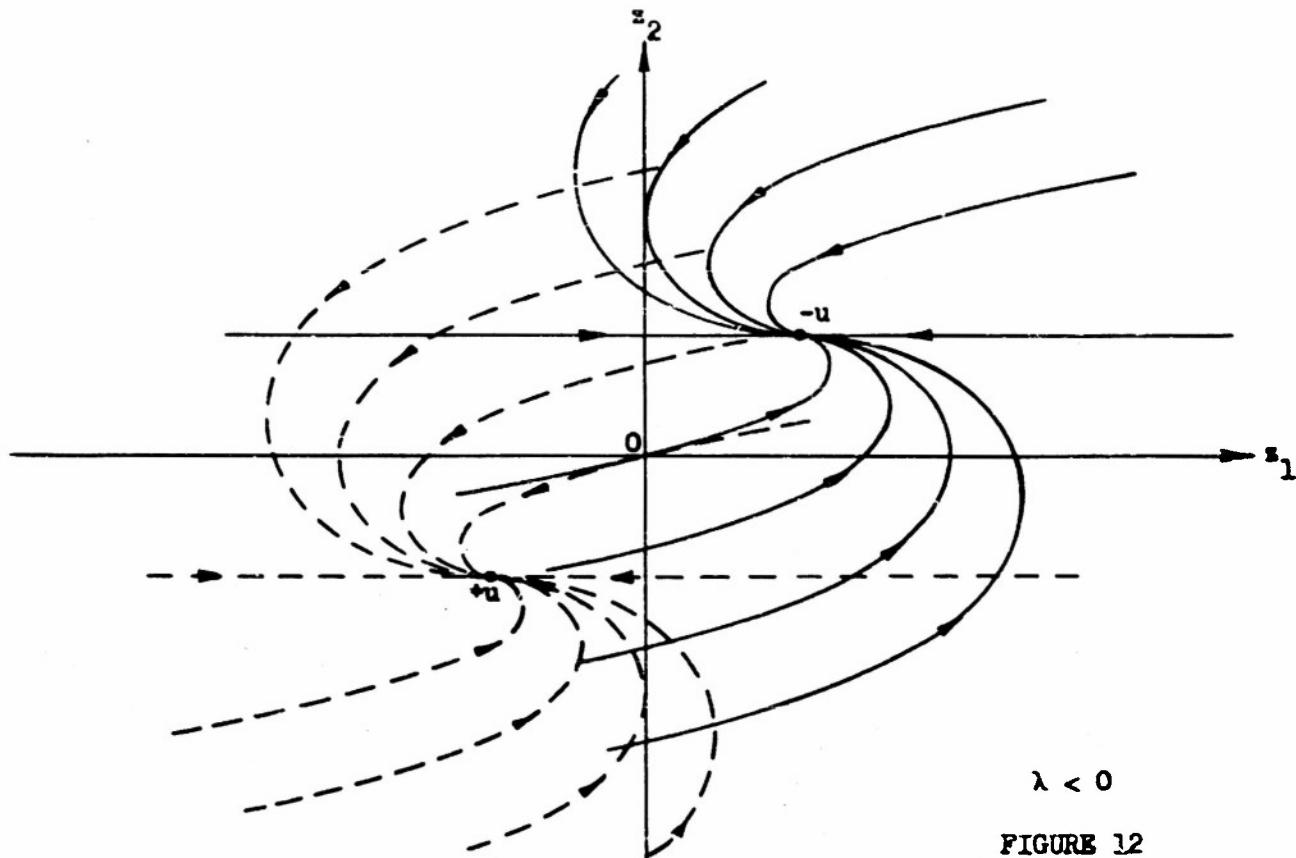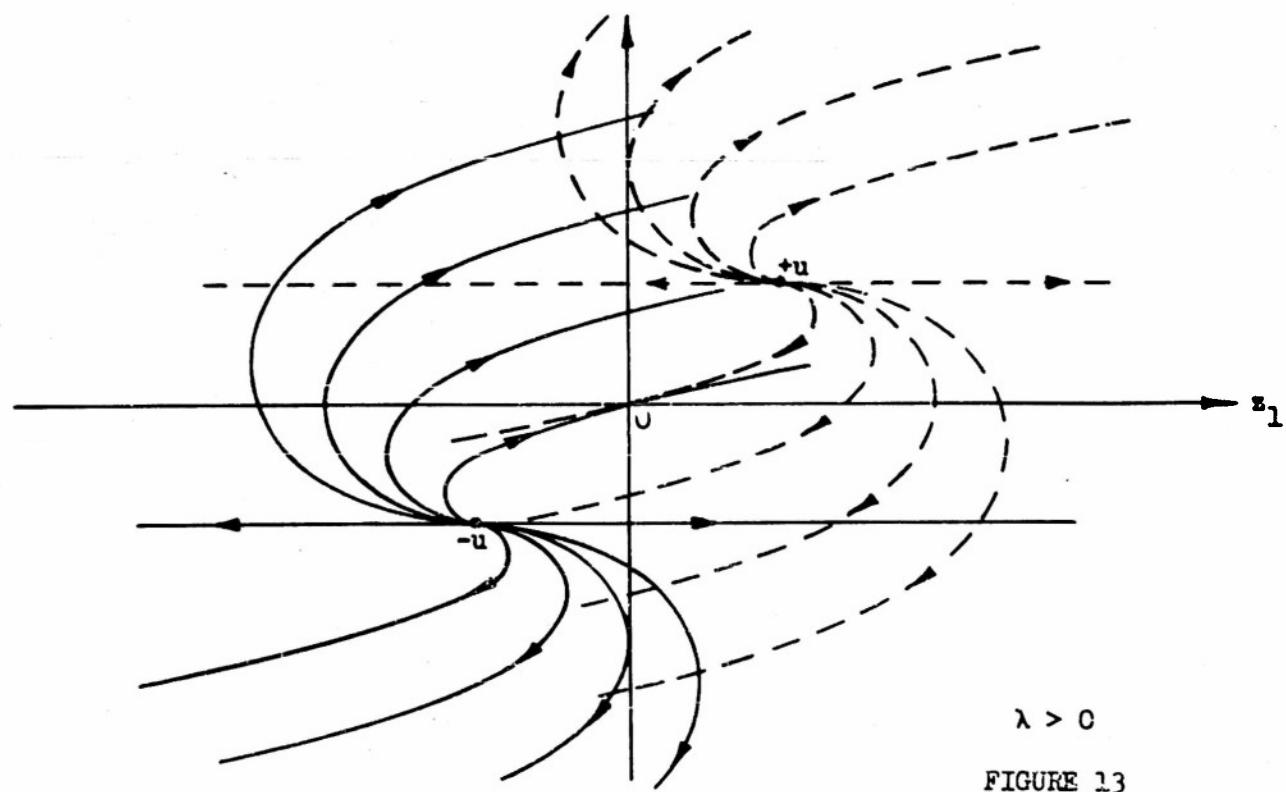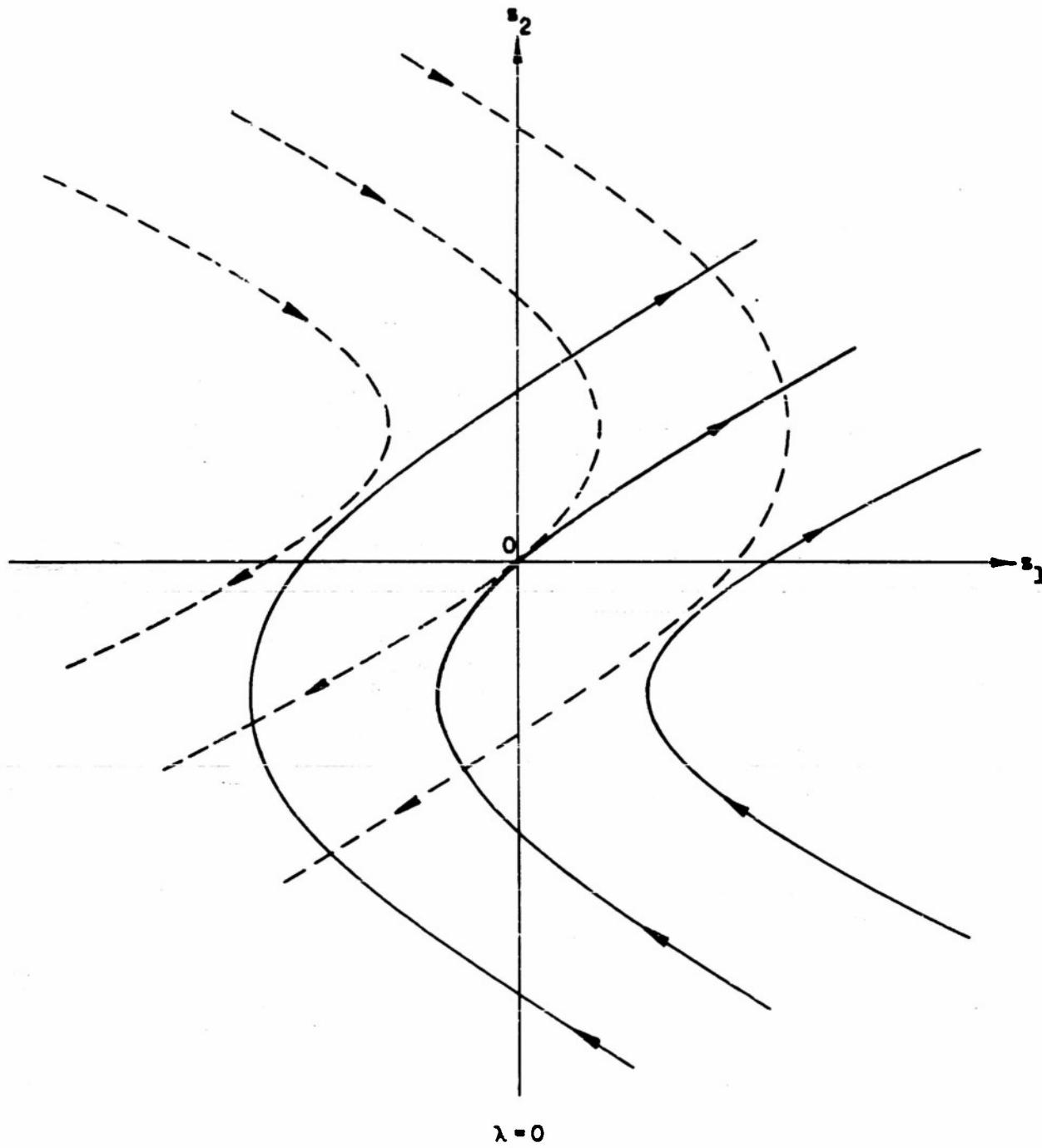
FIGURE 13

$\lambda = 0$

FIGURE 14

path exists for points in Q, and no paths exist for points outside of Q. For points in R^-, the minimal path consists of following the N-arc through the point until it intersects Γ^+ and then following Γ^+ to the origin. Minimal paths for points in R^+ are obtained by symmetry. (In terms of the control function f, this means that $f = -1$ in $R^- + \Gamma^-$ and $f = +1$ in $R^+ + \Gamma^+$.)

The curve C is the common boundary of the regions R^+ and R^-. It is on this curve alone that the minimizing control function changes sign or "switches" from the value $+1$ to the value -1 or vice versa. For this reason, C is called the switching curve.

The region Q from which minimal paths exist depends on the nature of the eigenvalues. In the degenerate case where the vector e is an eigenvector of A, the region Q degenerates into the curve C. Aside from this case, Q is a "two-dimensional" region. If the eigenvalues are nonpositive, Q is the entire plane. If both eigenvalues are positive,[*] Q is a bounded portion of the plane, the boundary curves being the P-curve through $+A^{-1}e$ and the N-curve through $-A^{-1}e$ (both followed backwards). If only one eigenvalue is positive while the other is nonpositive, Q is an unbounded portion of the plane between two parallel lines, these straight lines being solutions of the DE that are parallel to the eigenvector corresponding to the positive eigenvalue.

It should be remarked here that, in an (uncontrolled) physical system, positive eigenvalues mean some sort of instability. However, it has been shown above that, by the proper use of an "on-off" control, the system can be made stable, provided the error and its derivative remain in the region Q.

Also, another way of characterizing the minimal paths is as follows:

If any paths at all exist from a point, then a unique minimal path exists. The minimal path is the path with a minimum number of corners.

The Minimal Problem for $\ddot{x}_1 = f(x_1, \dot{x}_1)$ with $|f| \leq 1$

The broader minimal problem will be considered for the simple DE

$$\ddot{x}_1 = f(x_1, \dot{x}_1)$$

---

[*] Includes repeated eigenvalues.

or the equivalent system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = f(x_1, x_2) \quad , \qquad (58)$$

where $|f(x_1, x_2)| \leq 1$. It will be shown that the minimizing function is still the same function as the one found among the restricted class of functions capable of taking on only the values $\overset{+}{-}1$.

In order to guarantee solutions to (58), it is necessary to put some restrictions on f; however, f is still allowed to have some discontinuities so that it will include admissible functions of the type $f = \overset{+}{-}1$. It is recalled that in the case $f = \overset{+}{-}1$, the only admissible functions were those such that the solution curve from any point consisted of a finite number of alternating P- and N-arcs. This is equivalent to the requirement that any bounded region can be split up into a finite number of subregions, in each of which f is continuous.

In the broader problem (58), it is assumed that any bounded region can be split up into a finite number of subregions, in each of which f is continuous and satisfies a Lipshitz condition. However, f should be defined at every point in the plane. This assures a unique solution from every point in the plane if it is required that the solution curve be a continuous curve (in the phase plane).

Let Δ be the minimal path obtained previously when $f = \overset{+}{-}1$, and let π be any solution to (58) with the same initial point as Δ. Denote τ as the time length of a solution curve. With this notation, it is only necessary to prove the following theorem:

<u>Theorem 8:</u> $\tau(\Delta) \leq \tau(\pi)$.

It will be sufficient to prove the theorem for initial points in R^- and above the x_1-axis. The path Δ from such an initial point x^o is shown on the following page (the path $x^o q r 0$):
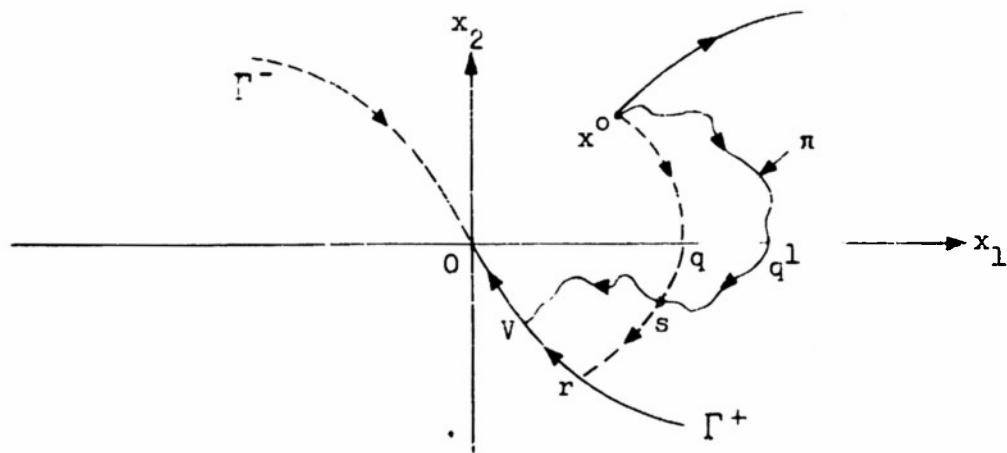
FIGURE 15

An inequality for any arc of π will now be obtained. Since $|f| \leq 1$,

$$-1 \leq \dot{x}_2 \leq +1 \quad . \tag{59}$$

Integrating this from 0 to t gives

$$-t \leq x_2 - x^0_2 \leq t \tag{60}$$

or

$$|t| \geq |x_2 - x^0_2| \qquad , \qquad \cdot \tag{61}$$

where the equality sign holds only for a P- or an N-arc. This expression proves the theorem immediately for points on Γ^+ or Γ^-.

An inequality for the slope of π is obtained next: Let the slope of π be m; then

$$|m| = \left| \frac{\dot{x}_2}{\dot{x}_1} \right| = \left| \frac{f}{x_2} \right| \leq \frac{1}{|x_2|} \tag{62}$$

or

$$- \frac{1}{|x_2|} \leq m \leq \frac{1}{|x_2|} \qquad . \qquad (63)$$

Therefore, the slope of π at any point lies between the slope of the N-curve and the slope of the P-curve through the point.

Consider the curve π through the point x^o. Because of (63), π cannot pass through either the P-curve through x^o or the N-arc $x^o q$ through x^o. If π is to reach the origin, it must therefore cut the x_1-axis at a point q^1 to the right of (or possibly at) the point q. From (61), it is seen that $\tau(x^o q) \leq \tau(x^o q^1)$.

Now consider the rest of π. If π passes below the horizontal line through r, (61) shows that $\tau(\Delta) \leq \tau(\pi)$, and the theorem is proved. Otherwise, if π is to reach the origin, it must intersect Δ at some point s along the arc qr. For this portion of π, it is seen again from (61) that $\tau(x^o qs) \leq \tau(x^o q^1 s)$. It is only necessary to prove a similar inequality for the remaining part of π. There are three possibilities to be considered:

1. π reaches the origin by staying entirely within the area bounded by the part of Δ, $qr0$, and the x_1-axis.

2. π leaves this region through the arc $r0$.

3. π leaves this region through the x_1-axis.

In case (1), $\tau(\Delta) \leq \tau(\pi)$ since the time length along any solution is

$$\int dt = \int \frac{dx_1}{\dot{x}_1} = \int \frac{dx_1}{x_2}$$

and $|x_2|$ along π is not greater than $|x_2|$ along Δ.

In case (2), suppose that π leaves the region for the first time at the point v on the arc $r0$. Then the desired inequality will hold up to v by case (1) and after v by equation (61) so that $\tau(\Delta) \leq \tau(\pi)$ will still hold.

In case (3), as soon as π gets to a point above the x_1-axis, the considerations discussed earlier will prevail and π will have to intersect the x_1-axis again to the right of its first intersection. Thus, the length of π has been made longer before case (1) or (2) applies, and the desired result follows.

It is fairly clear that the strict inequality will hold unless Δ is identical with π, but this means that f would be just the minimizing function previously found. This completes the proof.

N-DIMENSIONAL SYSTEMS

Introduction

Consider the minimal problem for the n-dimensional system

$$\dot{x} = Ax \overset{+}{-} e \qquad\qquad (e \neq 0) \quad , \tag{64}$$

where A is an n by n matrix with real eigenvalues. By an appropriate nonsingular substitution $x = Tz$, (64) can be reduced to

$$\dot{z} = Dz \overset{+}{-} w \qquad\qquad (w \neq 0) \quad , \tag{65}$$

where $w = T^{-1}e$ and $D = T^{-1}AT$, and D is in Jordan canonical form, that is,

$$D = \begin{bmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \bigcirc \\ & & \circ & \\ & & & \ddots & \\ \bigcirc & & & & \Lambda_k \end{bmatrix} . \tag{66}$$

Each of the Λ_i is a matrix having one of the following forms:

$$\text{Case (a):} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \bigcirc \\ & & \ddots & \\ & & & \ddots & \\ \bigcirc & & & & \lambda_j \end{bmatrix} \quad , $$

$$\text{Case (b):} \begin{bmatrix} \lambda & & & \\ & \lambda & & \bigcirc \\ & & \circ & \\ & & & \circ & \\ \bigcirc & & & & \lambda \end{bmatrix} \quad , \tag{67}$$

$$\text{Case (c): } \begin{bmatrix} \lambda & 1 & & & & \bigcirc \\ & \lambda & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ \bigcirc & & & & & \lambda & 1 \\ & & & & & & \lambda \end{bmatrix} \quad \cdot \qquad \left.\right\} \qquad (67)$$

Here, λ_i and λ are eigenvalues of A.

The equations of the P- and N-curves are still given by

$$z = \epsilon^{Dt} z^o \stackrel{+}{-} \epsilon^{Dt} \int_o^t \epsilon^{-Ds} \, ds \cdot w \qquad . \qquad (68)$$

The matrix function ϵ^{Dt} is

$$\epsilon^{Dt} = \begin{bmatrix} \epsilon^{\Lambda_1 t} & & & & \bigcirc \\ & \epsilon^{\Lambda_2 t} & & & \\ & & \cdot & & \\ \bigcirc & & & \cdot & \\ & & & & \epsilon^{\Lambda_k t} \end{bmatrix} \quad , \qquad (69)$$

where $\epsilon^{\Lambda_i t}$ has one of the three forms corresponding to cases (a), (b), (c) of (67):

$$\text{Case (a): } \begin{bmatrix} \epsilon^{\lambda_1 t} & & & & \bigcirc \\ & \epsilon^{\lambda_2 t} & & & \\ & & \cdot & & \\ \bigcirc & & & \cdot & \\ & & & & \epsilon^{\lambda_j t} \end{bmatrix} \quad , \qquad \left.\right\}$$

$$\text{Case (b): } \epsilon^{\lambda t} I \qquad , \qquad\qquad (70)$$

$$
\text{Case (c): } \epsilon^{\lambda t}
\begin{bmatrix}
1 & t & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{n-1}}{(n-1)!} \\
& 1 & t & & \dfrac{t^{n-2}}{(n-2)!} \\
& & \circ & & \circ \\
& & & \circ & \circ \\
& & & & \circ \\
& & & & 1 & t \\
& & & & & 1
\end{bmatrix}
\Biggr\} \qquad (70)
$$

The minimal problem will be solved first for the cases where D is equal to one of the matrices in (67), and then for the general case. Certain exceptional cases can occur and will be disposed of in the next section.

Degenerate Cases

As was seen in the two-dimensional case, certain degenerate cases can occur. The following theorem carries over directly from the two-dimensional case.

<u>Theorem 9:</u> <u>If</u> w <u>is an eigenvector of the matrix</u> D, <u>then minimal paths can exist only for points on</u> $\Gamma^+ + \Gamma^-$.

Γ^+ and Γ^- are defined just as in the two-dimensional case. The former proof holds good without change. It is recalled that Γ^+ and Γ^- lie on a common straight line through the origin.

Even if w is not an eigenvector of D, there still can occur degenerate cases. Suppose, for instance, that, in the system $\dot{z} = Dz + w$,

$$
D =
\begin{bmatrix}
\lambda_1 & 1 & 0 & & & \\
& \lambda_1 & 0 & & & \\
& & \lambda_2 & & & \\
& & & \circ & & \\
& & & & \circ & \\
& & & & & \lambda_{n-1}
\end{bmatrix}
\quad \text{and} \quad
w =
\begin{bmatrix}
2 \\
1 \\
0 \\
\circ \\
\circ \\
\circ \\
0
\end{bmatrix}
, \qquad (71)
$$

where the $\lambda_1, \ldots, \lambda_{n-1}$ are distinct. It is easily seen that w is not an eigenvector of D. However, consider the system

$$\dot{\bar{z}} = \bar{D}\bar{z} + \dot{\bar{w}} \quad ,$$

where $\bar{z}$ denotes the $n-1$ dimensional vector whose components are $z_2, z_3, \ldots, z_n$, and

$$\bar{D} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & \ddots & \\ & & & & \lambda_{n-1} \end{bmatrix} \quad \text{and} \quad \bar{w} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad . \tag{72}$$

It can be seen that $\bar{w}$ is an eigenvector of $\bar{D}$ corresponding to the eigenvalue λ_1. In this $n-1$ dimensional system, the equations of the P- and N-curves are obtained simply by leaving off the equation for z_1 in the n-dimensional system. If a path from $\bar{z}$ in the $n-1$ dimensional system is considered, a unique path to a unique point z in the n-dimensional system can be determined simply by working backwards from the origin and following P- and N-curves for the same lengths of time as for the path in the $n-1$ dimensional system. Conversely, every path from z will, in the same manner, determine a unique path from its projection $\bar{z}$. If no path exists from $\bar{z}$, no path will exist from z.

Now, according to Theorem 9, paths can exist in the $n-1$ dimensional system only for points on a certain straight line through the origin. Therefore, paths can exist in the n-dimensional space only for points which project onto this straight line. These are points with z_1 arbitrary and $z_2, z_3, \ldots, z_n$ satisfying an equation of the form

$$\frac{z_2 - w_2}{w_2} = \frac{z_3 - w_3}{w_3} = \ldots = \frac{z_n - w_n}{w_n} \quad .$$

Thus, minimal paths can exist only for a two-dimensional plane through the origin.

It is pointed out that the above demonstration depended on the fact that the equations for $z_2, z_3, \ldots, z_n$ did not contain z_1. If z_1 were contained, then $\bar{z}$ would not be a solution of the same type of differential equation as z.

Reasoning similar to the above will prove:

Theorem 10: Suppose the k-dimensional system

$$\dot{\bar{z}} = \bar{D}\bar{z} \overset{+}{-} \bar{w} \qquad (\bar{w} \neq 0)$$

can be formed by taking k of the component equations of the n-dimensional system

$$\dot{z} = Dz \overset{+}{-} w \qquad (w \neq 0) \quad .$$

If $\bar{w}$ is an eigenvector of $\bar{D}$, then minimal paths can exist only in a certain n-k+1 dimensional hyperplane through the origin.

In order to avoid these exceptional cases, it is assumed in the sequel that (1) if λ is a repeated eigenvalue, λ appears only in a diagonal matrix of the form (c) of (67) and in only one such matrix, and (2) $w_i \neq 0$ for $i = 1, 2, \ldots, n$. Such a system is called a nondegenerate system. Theorem 10 holds only for $k-1$ for a nondegenerate system, and, in fact, it will be seen that minimal paths will exist for all points in some n-dimensional neighborhood of the origin.

Terminology and Statement of the Main Theorem

In finding the minimal paths, it will again be convenient to work backwards from the origin. Using this procedure, the following point sets or "surfaces" in n-dimensional space are defined:

R_1: The point set obtained by following the P-curve through the origin backwards in time. R_1 is the set of points z^1 satisfying

$$z^1 = \epsilon^{-Dt_1} \int_0^{-t_1} \epsilon^{-Ds} \, ds \cdot w \qquad (t_1 > 0) \quad .$$

R_2: The point set obtained by starting at any point of R_1 and following the N-curve through the point backwards in time, that is, points z^2 such that

$$z^2 = \epsilon^{-Dt_2} z^1 - \epsilon^{-Dt_2} \int_0^{-t_2} \epsilon^{-Ds} \, ds \cdot w \qquad (t_2 > 0) \quad .$$

.
.
.

R_n: The points z^n satisfying

$$z^n = \epsilon^{-Dt_n} z^{n-1} + (-1)^{n-1} \epsilon^{-Dt_n} \int_0^{-t_n} \epsilon^{-Ds} \, ds \cdot w \qquad (t_n > 0) \quad .$$

Let R_k^{-1} be the point set obtained by reflecting R_k in the origin.

Let $Q_k = R_k + R_k^{-1}$.

Let $Q_k^* = \sum_{i=1}^k Q_i + 0$.

R_1 is the same as the curve Γ^+ previously defined and R_1^{-1} is the same as Γ^-. Q_1 consists of all points from which a path exists with no corners; Q_2 consists of all points from which a path exists with exactly one corner; and, in general, Q_k consists of all points from which a path exists with exactly $k-1$ corners. Q_k^* is the logical sum of the sets $Q_1, \ldots, Q_k$ and consists of all points from which paths exist with $k-1$ or less corners.

Two theorems will now be formulated. To prove these theorems, it is necessary to consider several cases which will be done in the following sections.

Theorem 11: In a nondegenerate system with real eigenvalues, the point sets $Q_1, Q_2, \ldots, Q_n$ are mutually exclusive. From each point in Q_k, a unique path exists with exactly $k-1$ corners, and no path exists with less than $k-1$ corners.

The main theorem of this report is:

Theorem 12: In a nondegenerate system with real eigenvalues, a unique minimal path exists from points in the region Q_n^*, and no paths exist outside of Q_n^*. The minimal path from a point in Q_k is the unique path with exactly $k-1$ corners.

As is seen from the above theorems, all minimal paths contain $n-1$ or less corners. All the corners of minimal paths lie on the surface Q_{n-1}^*, which is called the switching surface.

Distinct Eigenvalues

Consider the case where the matrix D in the DE

$$\dot{z} = Dz \overset{+}{-} w \qquad\qquad (w_i \neq 0) \tag{73}$$

has the form

$$D = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \bigcirc & \\ & & \cdot & & \\ & & & \cdot & \\ & \bigcirc & & \cdot & \\ & & & & \lambda_n \end{bmatrix} \qquad , \tag{74}$$

where the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct. It will be assumed, for the present, that no eigenvalue is zero. The equations of the P- and N-curves are then

$$z \overset{+}{-} u = \epsilon^{Dt} (z^0 \overset{+}{-} u) \quad , \tag{75}$$

where $u = D^{-1}w$ and $u_i \neq 0$ since $w_i \neq 0$.

Theorem 13: The sets $Q_1, Q_2, \ldots, Q_n$ are mutually exclusive. From each point in Q_k, a unique path exists with exactly $k-1$ corners, and no path exists with less than $k-1$ corners. No paths exist from points outside of Q_n^*.

This theorem will be proved by induction. The proof for $n = 2$ was given earlier. The idea of projection will be used again. Denote the

n-1 dimensional system obtained by leaving off the first equation in (73) by

$$\dot{\bar{z}} = \bar{D}\bar{z} \pm \bar{w} \qquad .$$ (76)

$\bar{Q}_k$ denotes the regions in the n-1 dimensional space with which the theorem is concerned. By the inductive assumption, the theorem holds for $\bar{Q}_1 , \bar{Q}_2 , \ldots , \bar{0}_{n-1}$. Consider any point $\bar{z}$ in $\bar{Q}_k$. A unique path with exactly k-1 corners exists from this point. This path consists of a definite sequence of P- and N-curves followed for definite times in the n-1 dimensional space. By starting at the origin and following the same sequence backwards for the same times, but now in n dimensions, a unique point z is reached by means of a path with exactly k-1 corners. This path is unique since otherwise the corresponding path from $\bar{z}$ would not be unique. There is no path from z with less than k-1 corners since no path from $\bar{z}$ contains less than k-1 corners. z is therefore in Q_k . Similar reasoning shows that to each point z in Q_k there is a unique point $\bar{z}$ in $\bar{Q}_k$; therefore, a unique path of exactly k-1 corners exists from z and no path with less than k-1 corners. This proves the theorem for the sets $Q_1 , Q_2 , \ldots , Q_{n-1}$. Incidentally, it can be seen from the above that a one-to-one correspondence exists between $\bar{Q}_1 , \bar{Q}_2 , \ldots , \bar{Q}_{n-1}$ and $Q_1 , Q_2 , \ldots , Q_{n-1}$. The switching surface

$$Q_{n-1}^* = \sum_{i=1}^{n-1} Q_i + 0 \qquad .$$

can therefore be represented as a single-valued function

$$z_1 = F(z_2 , z_3 , \ldots , z_n) \qquad .$$

From the definition of the Q-sets, it also follows that the switching surface is continuous.

The theorem for the set Q_n has not yet been proved. To do this, it is necessary to have the explicit expression for points in Q_n . It is recalled that $Q_n = R_n + R_n^{-1}$. The equation for points in R_n is ob-

tained from the definition of the h-sets and by the process of successive elimination. If z^n represents any point in R_n, then

$$z^n + (-1)^{n-1}u = \left\{ \epsilon^{-D(\tau_1 + \tau_2 + \ldots + \tau_n)} \right.$$
$$\left. -2\epsilon^{-D(\tau_2 + \ldots + \tau_n)} + \ldots + (-1)^{n-1}2\epsilon^{-D\tau_n} \right\} u \quad , \quad (77)$$

where all the τ_i are greater than zero. The equation for points in R_n^{-1} is obtained by replacing z^n by $-z^n$. Parameters are changed by the nonsingular substitution

$$\left. \begin{array}{l} a_1 = \tau_1 + \tau_2 + \ldots + \tau_n \\ a_2 = \tau_2 + \ldots + \tau_n \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} = \tau_{n-1} + \tau_n \\ a_n = \tau_n \end{array} \right\} \quad (78)$$

It is noted that $a_1 > a_2 > \ldots > a_n > 0$. Equation (77) now becomes

$$z^n + (-1)^{n-1}u = \left\{ \epsilon^{-Da_1} - 2\epsilon^{-Da_2} + \ldots + (-1)^{n-1}2\epsilon^{-Da_n} \right\} u \quad . \quad (79)$$

Now if z^n is a point in Q_{n-1}, equations (79) are satisfied by certain values of $a_1, a_2, \ldots, a_{n-1}$ and $a_n = 0$. Consider the Jacobian of this system:

$$\left| \frac{\partial z_i^n}{\partial a_j} \right| = K \prod_{i=1}^{n} (\lambda_i u_i) \begin{vmatrix} \epsilon^{\lambda_1 a_1} & \epsilon^{\lambda_1 a_2} \ldots \epsilon^{\lambda_1 a_n} \\ \epsilon^{\lambda_2 a_1} & \epsilon^{\lambda_2 a_2} \ldots \epsilon^{\lambda_2 a_n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \epsilon^{\lambda_n a_1} & \epsilon^{\lambda_n a_2} \ldots \epsilon^{\lambda_n a_n} \end{vmatrix} \quad . \quad (80)$$

Here, K is a nonzero constant. The λ_i are all different from zero by assumption, and the u_i are all different from zero since the system is nondegenerate. The determinant is also different from zero if the λ_i and the a_i are distinct, which is the case here. (The determinant is a generalization of the Vandermonde determinant, and a proof that it does not vanish is given in Reference 3, part V, problem 26.) Equation (79) can therefore be solved uniquely for $a_1, a_2, \ldots, a_n$ in some neighborhood of z^n. This shows that, in some neighborhood of z^n, a unique path with exactly n-1 corners exists.* As a_n is increased, the Jacobian remains different from zero so that a unique path with exactly n-1 corners exists from every point in Q_n. With the same type of reasoning, it can be established that no path exists from a point in Q_n with less than n-1 corners.

To complete the proof, it must be shown that no paths exist from points outside of Q_n^*. Consider such a point z. If a path exists from z, it must contain more than n-1 corners; say it contains m-1 corners with $m > n$. An equation similar to (79) could be set up containing m parameters. These equations could always be solved for the first n of these parameters and the other parameters set equal to zero. Therefore, a path exists from z with n-1 or less corners and z must therefore be in Q_n^*. This completes the proof.

A word should be said here about the extent of Q_n^*, the region of existence of paths. From (79), it is seen that, if some of the eigenvalues are negative, the region Q_n^* will extend to infinity in some directions, while if the eigenvalues are all positive, the region Q_n^* will be bounded.

More precise information is possible. It could be proved that, if the eigenvalues are all negative, paths exist from all points, while if the eigenvalue λ_i is positive, paths do not exist for points $|z_i| \geq |w_i/\lambda_i|$. In the latter case, paths do not necessarily exist from all points satisfying $|z_i| < |w_i/\lambda_i|$.

---

* If it happened that this solution gave $a_n = 0$ for a point in the neighborhood of z^n, then the point would certainly be in Q_{n-1} and could not be in Q_n because of the uniqueness of the solution.

The main theorem for this case will now be proved.

Theorem 14: If the eigenvalues are distinct and not zero, a unique minimal path exists for all points in Q_n^*. The minimal path for a point in Q_k is the unique path that contains k-1 corners.

This theorem will be proved by induction. The theorem has been proved for n = 2.

The theorem will be proved first for points on the switching surface Q_{n-1}^*. Consider any point z in Q_k for $k \leq n-1$ and its projection $\bar{z}$ in $\bar{Q}_k$ as in the proof of the preceding theorem. The unique path Δ_{k-1} with k-1 corners from z projects onto the unique path $\bar{\Delta}_{k-1}$ with k-1 corners from $\bar{z}$. The paths Δ_{k-1} and $\bar{\Delta}_{k-1}$ have the same time length.* In the n-1 dimensional space, $\bar{\Delta}_{k-1}$ is the minimal path by assumption. Any path from z other than Δ_{k-1} would project into a path different from $\bar{\Delta}_{k-1}$ and would therefore have a longer time length than Δ_{k-1}. Therefore, Δ_{k-1} is the minimal path and the theorem is proved for all points in Q_{n-1}^*.

It remains to prove the theorem for points in Q_n. Suppose z is any point in Q_n. It is known that there exists a unique path Δ_{n-1} from z containing exactly n-1 corners; it is desired to show that Δ_{n-1} is the minimal path.

Consider any path at all from z. This path will have a certain number of corners and will then intersect Q_{n-1}. From this point, the minimal path is uniquely determined. It is only necessary to determine the behavior of the minimal path from z until its first intersection with Q_{n-1}. For convenience, label the point z as z^{n+1}. Consider the path Δ_n from z^{n+1} that has exactly one corner before reaching Q_{n-1}. In other words, Δ_n is the path obtained by starting at z^{n+1} and following, say, a P-arc for a time $\tau_n > 0$ to a point z^n, not in Q_{n-1}, and then an N-arc, for a time $\tau_{n-1} > 0$ to a point z^{n-1}, which is in Q_{n-1}. From this point, the minimal path is determined. It will be shown that the time length of Δ_n can be reduced by making τ_n smaller,

---

* Any path and its "projected" path have the same time length by construction.

and that it is a minimum if $\tau_n = 0$. This will show that the time length of Δ_{n-1} is less than the time length of Δ_n. Successive use of this result will show that Δ_{n-1} is shorter than a path with any number of corners.

Using the equations of the P- and N-curves, the equations for the successive corners of Δ_n are found to be

$$
\left.
\begin{aligned}
z^{n+1} + u &= \epsilon^{-D(\tau_n)}(z^n + u) \\[6pt]
z^n - u &= \epsilon^{-D\tau_{n-1}}(z^{n-1} - u) \\[6pt]
z^{n-1} + u &= \epsilon^{-D\tau_{n-2}}(z^{n-2} + u) \\[2pt]
&\;\;\vdots \\[6pt]
z^1 + (-1)^{n+1}u &= \epsilon^{-D\tau_0}\left(z^0 + (-1)^{n+1}u\right) \qquad (z^0 = 0,\ \tau_i > 0)
\end{aligned}
\right\} \quad (81)
$$

By successive elimination, it is found that

$$
z^{n+1} + (-1)^{n+1}u = \left\{ 2\epsilon^{-D\tau_n} - 2\epsilon^{-D(\tau_n + \tau_{n-1})} + \ldots + (-1)^{n-1}2\epsilon^{-D(\tau_1 + \ldots + \tau_n)} \right.
$$
$$
\left. + (-1)^n \epsilon^{-D(\tau_0 + \ldots + \tau_n)} \right\} u \quad . \tag{82}
$$

Now let

$$
\left.
\begin{aligned}
a_0 &= \tau_0 + \tau_1 + \ldots + \tau_n \\
a_1 &= \qquad \tau_1 + \ldots + \tau_n \\
&\;\;\vdots \qquad\qquad\qquad \vdots \\
a_{n-1} &= \qquad\qquad\qquad \tau_{n-1} + \tau_n \\
a_n &= \qquad\qquad\qquad\qquad \tau_n
\end{aligned}
\right\} \quad (83)
$$

where a_c represents the total time length of Δ_n,

Substituting (83) into (82) gives

$$z^{n+1} + (-1)^{n+1}u = \left\{ 2\epsilon^{-Da_n} - 2\epsilon^{-Da_{n-1}} + \ldots + (-1)^{n-1}2\epsilon^{-Da_1} \right.$$
$$\left. + (-1)^n \epsilon^{-Da_o} \right\} u \tag{84}$$

or, taking components,

$$z_i^{n+1} + (-1)^{n+1}u_i = \left\{ 2\epsilon^{-\lambda_i a_n} - 2\epsilon^{-\lambda_i a_{n-1}} + \ldots + (-1)^n \epsilon^{-\lambda_i a_o} \right\} u_i \ . \tag{85}$$

Let $\tau_n = a_n$ be the independent variable; then differentiating both sides of (85) with respect to a_n results in

$$0 = 2\epsilon^{-\lambda_i a_n} - 2\epsilon^{-\lambda_i a_{n-1}} \dot{a}_{n-1} + \ldots + (-1)^{n-1}2\epsilon^{-\lambda_i a_1} \dot{a}_1$$
$$+ (-1)^n \epsilon^{-\lambda_i a_o} \dot{a}_o \qquad (i = 1,\ldots,n) \quad , \tag{86}$$

where $\dot{a}_i = da_i/da_n$. These are n simultaneous equations for $\dot{a}_o, \dot{a}_1, \ldots, \dot{a}_{n-1}$. Solving by determinants for $\dot{a}_o$ gives

$$\frac{\dot{a}_o}{2} = \frac{\begin{vmatrix} \epsilon^{-\lambda_1 a_n} & \epsilon^{-\lambda_1 a_{n-1}} & . & . & . & \epsilon^{-\lambda_1 a_1} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \epsilon^{-\lambda_n a_n} & \epsilon^{-\lambda_n a_{n-1}} & . & . & . & \epsilon^{-\lambda_n a_1} \end{vmatrix}}{\begin{vmatrix} \epsilon^{-\lambda_1 a_{n-1}} & . & . & . & . & \epsilon^{-\lambda_1 a_o} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \epsilon^{-\lambda_n a_{n-1}} & . & . & . & . & \epsilon^{-\lambda_n a_o} \end{vmatrix}} \ . \tag{87}$$

The determinant in the denominator of (87) is the same as the one previously encountered, and is different from zero. Since the numerator is of the same form as the denominator, the quotient is positive. This shows that $\dot{a}_o > 0$ when $\tau_n > 0$, and therefore the minimum occurs when $\tau_n = 0$, as was to be proved.

If one of the eigenvalues is zero, say the last one, nothing is changed in the above work except the equation for z_n. Since the DE for $\dot{z}_n$ is

$$\dot{z}_n = \overset{+}{-} w_n \qquad , \tag{88}$$

the solution is

$$z_n = \overset{+}{-} w_n t + z^o_n \quad . \tag{89}$$

It is noted that if the equation in the case of distinct eigenvalues is taken for z_n, namely,

$$z_n \overset{+}{-} \frac{w_n}{\lambda_n} = \epsilon^{\lambda_n t} \left(z^o_n \overset{+}{-} \frac{w_n}{\lambda_n} \right) \qquad ,$$

and the limit is taken as $\lambda_n \to 0$, equation (89) is obtained. Thus, it is not necessary to go through all the algebra of successive substitution. All that is necessary is to let $\lambda_n \to 0$ everywhere. This, however, does not cause the determinants used in the proofs of the theorems to vanish. Thus, the theorems are true if one eigenvalue is zero, and are therefore true for all cases where the eigenvalues are distinct.

Repeated Eigenvalues

Consider the case where D, is of the form

$$D = \begin{bmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & \cdot & \cdot \\ & & & & & & \lambda & 1 \\ & & & & & & & \lambda \end{bmatrix} \qquad . \tag{90}$$

For the present, it is assumed that $\lambda \neq 0$. The P- and N-curves are given by

$$z \overset{+}{-} u = \epsilon^{Dt}(z^o \overset{+}{-} u) \quad , \tag{91}$$

where

$$\epsilon^{Dt} = \epsilon^{\lambda t} \begin{bmatrix} 1 & t & \dfrac{t^2}{2} & \cdots & \cdots & \dfrac{t^{n-1}}{(n-1)!} \\ & 1 & t & & & \dfrac{t^{n-2}}{(n-2)!} \\ & & & \cdot & & \cdot \\ & & & & \cdot & \cdot \\ & & & & \cdot & \cdot \\ & & & & 1 & t \\ & & & & & 1 \end{bmatrix} \quad . \tag{92}$$

The solution of the minimal problem will follow the same lines as in the last section and most of the proof does not need to be repeated. In fact, all that is necessary is to verify that the two determinants which correspond to (80) and (87) are different from zero.

Consider the determinant corresponding to (87). Equations (84) for the path Δ_n still hold, namely,

$$z^{n+1} + (-1)^{n+1}u = \left\{ 2\epsilon^{-Da_n} + \ldots + (-1)^{n-1}2\epsilon^{-Da_1} \right. $$
$$\left. + (-1)^n\epsilon^{-Da_o} \right\}u \quad . \tag{93}$$

Differentiating both sides of this vector equation with respect to a_n gives

$$0 = -D\left\{ 2\epsilon^{-Da_n}u - 2\epsilon^{-Da_{n-1}}u\dot{a}_{n-1} + \ldots + (-1)^{n-1}2\epsilon^{-Da_1}u\dot{a}_1 \right. $$
$$\left. + (-1)^n\epsilon^{-Da_o}u\dot{a}_o \right\} \quad . \tag{94}$$

Since D is nonsingular, the vector in braces must be zero. Taking the
n-component equations of (94) and solving for $\dot{a}_o$, gives, after some
algebra,

$$\frac{\dot{a}_o}{2} = \frac{\begin{vmatrix} \epsilon^{-\lambda a_n} \begin{matrix} \displaystyle\sum_{0}^{n-1} \frac{(-a_n)^k}{k!} u_{k+1} & \circ & \bullet & \circ & \displaystyle\sum_{0}^{n-1} \frac{(-a_1)^k}{k!} u_{k+1} \\ \displaystyle\sum_{0}^{n-2} \frac{(-a_n)^k}{k!} u_{k+2} & \circ & \circ & \circ & \displaystyle\sum_{0}^{n-2} \frac{(-a_1)^k}{k!} u_{k+2} \\ \vdots & & & & \vdots \\ u_{n-1} - a_n u_n & \circ & \bullet & \circ & u_{n-1} - a_1 u_n \\ u_n & \bullet & \circ & \bullet & u_n \end{matrix} \end{vmatrix}}{\begin{vmatrix} \epsilon^{-\lambda a_o} \begin{matrix} \displaystyle\sum_{0}^{n-1} \frac{(-a_{n-1})^k}{k!} u_{k+1} & \bullet & \circ & \circ & \displaystyle\sum_{0}^{n-1} \frac{(-a_o)^k}{k!} u_{k+1} \\ \vdots & & & & \vdots \\ u_n & \circ & \circ & \circ & u_n \end{matrix} \end{vmatrix}} \quad \cdot (95)$$

Fortunately, these determinants can be vastly simplified. Per-
forming obvious elementary operations on rows, starting from the nth
row and working upwards, results in

$$\frac{\dot{a}_0}{2} = \frac{\epsilon^{-\lambda a_n} \begin{vmatrix} a_n^{n-1} & a_{n-1}^{n-1} & \cdot & \cdot & \cdot & a_1^{n-1} \\ a_n^{n-2} & a_{n-1}^{n-2} & & & & a_1^{n-2} \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ a_n & a_{n-1} & \cdot & \cdot & \cdot & a_1 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 \end{vmatrix}}{\epsilon^{-\lambda a_0} \begin{vmatrix} a_{n-1}^{n-1} & \cdot & \cdot & \cdot & \cdot & a_0^{n-1} \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ a_{n-1} & & & & & a_0 \\ 1 & \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix}} \qquad (96)$$

The determinants appearing are the well-known Vandermonde determinants and are different from zero if the elements a_i are distinct, which is the case here. Since the numerator and the denominator have the same form, the quotient is positive, as was to be proved.

In the same way, the determinant that corresponds to (80) can be shown to be different from zero. In fact, it turns out to be the Vandermonde determinant again.

Therefore, the main theorem holds when the matrix D is the elementary Jordan matrix (90) and the eigenvalue is different from zero.

If the eigenvalue in (90) is zero, then

$$D = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & \bigcirc & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \bigcirc & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix} \qquad (97)$$

The equations of the P- and N-curves can be obtained easily by solving $\dot{z} = Dz \pm w$ in the ordinary manner. They are

$$
\left.
\begin{aligned}
z_1 &= \pm \left(w_1 t + w_2 \frac{t^2}{2!} + \ldots + w_n \frac{t^n}{n!} \right) + \left\{ z^o_1 + z^o_2 t + \ldots + z^o_n \frac{t^{n-1}}{(n-1)!} \right\} \\
&\quad \vdots \\
z_{n-1} &= \pm \left(w_{n-1} t + w_n \frac{t^2}{2!} \right) + (z^o_{n-1} + z^o_n t) \\
z_n &= \pm w_n t + z^o_n
\end{aligned}
\right\} \quad (98)
$$

Again, it turns out that (98) can be obtained from the solutions (91), where $\lambda \neq 0$, simply by taking the limit as $\lambda \to 0$. Since the determinants in (96) do not involve λ, the determinants obtained for $\lambda = 0$ will be exactly the same. Thus, the main theorem also holds for an elementary Jordan matrix when the eigenvalue is zero.

The General Case

Suppose the matrix D has the form

$$
D = \begin{bmatrix} \Lambda_1 & & & & \\ & \Lambda_2 & & \bigcirc & \\ & & \ddots & & \\ & & & \ddots & \\ & \bigcirc & & & \Lambda_k \end{bmatrix} , \quad (99)
$$

where each of the Λ_i is an elementary Jordan matrix of the form

$$
\Lambda_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & \bigcirc & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & \bigcirc & & & \lambda_i & 1 \\ & & & & & \lambda_i \end{bmatrix} . \quad (100)
$$

The matrices Λ_i can be one by one matrices so that purely diagonal boxes can be included in D.

The system

$$\dot{z} = Dz \overset{+}{-} w \tag{101}$$

will be nondegenerate if $w_i \neq 0$, $i = 1,2,\ldots,n$ and if an eigenvalue λ_i appears in only <u>one</u> matrix of the form (100).

To prove that the main theorem holds in this general case, it is only necessary, again, to consider the determinants corresponding to the two determinants used before. In order to avoid too many notational difficulties, the following special case is considered:

$$D = \begin{bmatrix} \Lambda_I & 0 & 0 \\ 0 & \Lambda_{II} & 0 \\ 0 & 0 & \Lambda_{III} \end{bmatrix} \quad , \tag{102}$$

where

$$\tag{103}$$

It is assumed that Λ_I is of order k_1, Λ_{II} of order k_2, and Λ_{III} of order k_3. It is also assumed that $\lambda_1, \ldots, \lambda_n, \lambda$ are distinct and not zero (this is necessary for the system to be nondegenerate).

Let z_i and w be partitioned as follows:

$$z = \begin{bmatrix} z_I \\ z_{II} \\ z_{III} \end{bmatrix} , \qquad w = \begin{bmatrix} w_I \\ w_{II} \\ w_{III} \end{bmatrix} , \tag{104}$$

where z_I is a k_1-dimensional, z_{II} a k_2-dimensional, and z_{III} a k_3-dimensional vector, and the same for w_I, w_{II}, w_{III}.

The DE's under consideration can be written as

$$\dot{z}_i = \Lambda_i z_i \stackrel{+}{-} w_i \qquad (i = I, II, III) . \tag{105}$$

These are the same type of matrix equations that have been solved previously. After a little reflection, the relevant determinant can be written without further calculation. The denominator of the determinant used in the proof of the main theorem is (106) below:

$$\begin{vmatrix} \epsilon^{-\lambda_1 a_{n-1}} & \cdots & \epsilon^{-\lambda_1 a_0} \\ \vdots & & \vdots \\ \epsilon^{-\lambda_{k_1} a_{n-1}} & \cdots & \epsilon^{-\lambda_{k_1} a_0} \\ \epsilon^{-\lambda a_{n-1}}(a_{n-1})^{k_2-1} & \cdots & \epsilon^{-\lambda a_0} a_0^{k_2-1} \\ \vdots & & \vdots \\ \epsilon^{-\lambda a_{n-1}} & \cdots & \epsilon^{-\lambda a_0} \\ a_{n-1}^{k_3-1} & \cdots & a_0^{k_3-1} \\ \vdots & & \vdots \\ a_{n-1}^2 & \cdots & a_0^2 \\ a_{n-1} & \cdots & a_0 \\ 1 & \cdots & 1 \end{vmatrix} \tag{106}$$

The only comment that is necessary concerns the appearance of such factors as $\epsilon^{-\lambda a_{n-1}}$ which did not appear when the special case of the matrix Λ_{II} was considered previously. The reason is that, previously, such common factors of each column of the determinant could be taken out; this cannot be done in the general case.

The fact that this determinant is different from zero follows from problem 75 and the solution to problem 76 in part V of Reference 3. The rest of the arguments given previously apply without change, so that the main theorem remains true in this case.

Clearly, any number of elementary Jordan matrices appearing in D will yield to the same treatment. Therefore, the main theorem* holds true for any nondegenerate system!

Statement of Results

The original problem posed has now been solved when the matrix of the system has real eigenvalues. The result can be stated quite simply as follows:

Theorem 15: Consider the n-dimensional system

$$\dot{x} = Ax \pm e \qquad (e \neq 0) \quad ,$$

where A has real eigenvalues. If a path exists from any point in the phase space, then a unique minimal path exists and is the path that contains a minimum number of corners.

The above theorem holds equally well for degenerate or nondegenerate systems. The only difference is in the region of existence of minimal paths.

For a nondegenerate system, minimal paths exist for all points in some n-dimensional neighborhood of the origin. If the eigenvalues are all nonpositive, minimal paths exist from all points in the space. If the eigenvalues are all positive, minimal paths exist in a bounded neighborhood of the origin. If some of the eigenvalues are positive and some nonpositive, minimal paths exist in an unbounded region lying between pairs of parallel hyperplanes.

---

* Theorem 12.

For a degenerate system, all the minimal paths lie on some hyperplane through the origin. The remarks in the preceding paragraph hold for this hyperplane.

CONCLUDING REMARKS

Some matters which are pertinent to practical applications of the results are briefly discussed below:

If a control system is to be designed for a minimum time of correction of an error and its derivatives, the results in this report and those of Reference 1 show that the best control is a properly designed limit control or bang-bang control. The mathematical features of the design of such a control have been demonstrated in many cases.

In the practical design of such a system, although the control mechanism itself is as simple as possible (a switch or relay), a computing device must be used along with it to instruct the control mechanism when to switch from one value to another. Before such computers are designed, a great deal of numerical computation must be done; however, the computations are not too complex. They would involve only the tabulation of the solution of certain linear differential equations.

It should be mentioned that a servo-motor control system of the type discussed has already been built (see Reference 4). The differential equation of the system was one of the simpler ones, namely, $\ddot{x} = \pm 1$. The system was found to give excellent results and to be much lighter and occupy less volume than a standard type of system under the same power conditions.

It should also be mentioned that the design of a limit control system depends on the parameters of the system to be controlled. However, it certainly cannot be expected that a control can be designed that will give optimum performance without taking into account the particular parameters of the system to be controlled.

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