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THE EFFECTS OF ROTOR BLADE FLEXIBILITY AND UNBALANCE on HELICOPTER HOVERING STABILITY & CONTROL

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THE EFFECTS OF ROTOR BLADE FLEXIBILITY AND UNBALANCE ON HELICOPTER HOVERING STABILITY AND CONTROL

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1. SUMMARY

A method is developed to determine the control response of a single-rotor helicopter equipped with unbalanced, flexible blades. The detailed analysis is simplified, without great loss in accuracy, in order to present a practical method for use by the helicopter engineer. The detailed analysis is included for those interested.

By use of this method, the influence of rotor blade flexibility and unbalance on the stability and control of a conventional aircraft is investigated. Typical values are assumed and their effects evaluated. It is shown that blade flexibility and unbalance can have a noticeable influence on the damping in pitch of a helicopter. Consequently, the period and divergence of the unstable fuselage oscillation, as well as the automatic control system requirements for stability, are affected. Neglect of blade flexibility and unbalance could materially reduce the effectiveness of an automatic control system design.

2. INTRODUCTION

To date, most treatises on helicopter stability and control have dealt with the problem under an assumption that the rotor blades are structurally rigid. This was necessitated, principally, for the sake of simplicity in the formulation of a basic theory. For contrary to this assumption, it is well known that in practice, helicopter blades have a high degree of flexibility and are subjected to deflections by the actions of various inertia and aerodynamic forces encountered in flight. Basic theories have now been established and refined (i.e. Ref. 1, 2, and 3), and are commonly accepted as capable of predicting the nature of helicopter stability and control. Therefore, it appears appropriate at this time, to extend the basic theory in order to study the effects of various existing conditions such as rotor blade flexibility and unbalance.

The motion of a conventional articulated or "see-saw" rotor blade, with reference to the rotor shaft, may be described as:

(1) Rotation about the rotor axis

(2) Oscillation about a flapping hinge

(3) Oscillation about a blade root feathering axis

(4) Structural twist about a blade span axis (elastic axis). This axis may coincide with the feathering axis, depending upon the structural characteristics of the rotor system.

Additional components of the motion are, oscillation as a pendulum in the plane of the rotor disk about a drag hinge (lagging), and blade structural bending in the flapping plane. These additional components can materially effect the twisting tendency (unbalance) of the blades. However, as was pointed out in Ref. 4, experimental evidence has shown that for articulated blades, these components directly have only a second-order influence on the air forces on the blade. This may not be the case for rotors in which the blades are rigidly connected to the shaft, or have blades of extremely low bending stiffness. In these instances, structural deformations (bending) in the flapping plane may be of prime importance (Ref. 6). The analysis presented herein is principally concerned with helicopters having an articulated or "see-saw" type rotor.

On the other hand, it has been shown by Wheatley (Ref. 4 and 5), that the torsional flexibility and unbalance of a rotor blade has a pronounced influence on the flapping motion of a blade. Except in the case of balanced blades (i.e. blades having symmetric airfoils whose elastic, aerodynamic, and mass gravity axes coincide), the resultant of the air forces and mass reactions produce a couple which tends to twist the blade. This resulting twist has a constant magnitude for pure hovering flight, while in disturbed or forward flight it has periodic values as well. This twist, naturally, affects the flapping motion of the blade, and consequently the characteristics of the helicopter's motion.

The existence of the above phenomenon has long been appreciated by blade designers, due to its associated disturbing effect on stick forces. To eliminate blade twist and the disturbing stick forces they have striven to design balanced blades. With this condition satisfied the disturbing stick forces due to the blades, as well as blade twist would be eliminated.

For it is well known that in the conventional rotor-control system, oscillating stick forces are due entirely to blade unbalance, having both mass and aerodynamic origins. Unfortunately, however, the attainment of balanced blades in practice is extremely difficult. This fact is readily apparent when one considers, as previously stated, blade distortions in flight, in the flapping and lag planes. In many cases a balanced blade design proves impractical. In addition, constructional imperfections and use often cause unbalance. Consequently, blade twist and oscillating stick forces are present in varying degrees in helicopters.

Due to the disturbing nature of these oscillating stick forces recent helicopter designs resort to the use of irreversible control systems. This approach tends to eliminate the disturbing stick forces. However, since the cause, blade unbalance, has not been eliminated, blade twist, and its associated effects are still present in these designs.

Contrary to the above tendency to eliminate automatic blade twist, there is a feeling among many that effective stability and control may be obtained by proper design of the rotor system utilizing unbalanced blades. Naturally, such a system would entail a minimum weight penalty on the already relatively poor carrying capacity of helicopters. For unlike the fixed-wing aircraft where automatic stability and control requires the installation of special gyroscopes, the helicopter can make use of its rotor system which itself is a gyroscope.

The results which follow attempt to give some indication as to the effects of blade flexibility and unbalance on the stability and control of a typical helicopter. The effects on the automatic control system requirements for stability are discussed in Section 7.

3. Results of Analysis

Similar to fixed-wing aeroelastic phenomena, the effect of blade flexibility depends primarily upon the relationship in flight, between the blade's elastic, aerodynamic and mass gravity axes. Using the analysis presented in Section 5, both mass and aerodynamically unbalanced blades have been investigated. To demonstrate the procedure and application of the analysis, a typical calculation is presented in Section 6.

For the quantitative investigation, the parameters of a typical 5000 pound helicopter were assumed (section 6) and the results compared to that obtained under the assumption of structurally rigid blades. The blades were considered fixed in torsion at their root end, and to be of uniform spanwise construction, thus having principal torsion modes of the type $\phi(r) = \sin \frac{\pi r}{2R}$. It is worthy to mention however, that in the analysis (Section 5) blade twist is introduced through a general torsion mode, $\phi(r)$, and consequently the analysis may be applied to helicopters with blades having any arbitrary torsional stiffness or mass distribution.

a) Mass Unbalanced Blades

For overbalanced (center of gravity forward of the elastic axis) as well as underbalanced (c.g. rear of elastic axis) blades the controlling dimensionless flexibility parameter was found to be $\left(\frac{I_2 \Omega^2}{K}\right)$; where, I_2 is the blade mass product of inertia about the flapping hinge and blade elastic axes (positive for forward c.g.), Ω is the rotor speed, and K is the blade torsional stiffness.

The principal effect of overbalanced blades is to increase the damping in pitch of the helicopter. This is shown in Fig. 1, in which the increase in damping in pitch of the helicopter is given for various values of $\left(\frac{I_2 \Omega^2}{K}\right)$ relative to the damping for $\left(\frac{I_2 \Omega^2}{K}\right) = 0$. This effect can be explained as follows. With conventional rotor-control systems, the rotor tip-path plane tends to pitch with the helicopter fuselage. However, in so doing, coriolis forces which oppose pitching of the tip-path plane are created on the blades. These forces are proportional to the pitching velocity of the tip-path plane, which is approximately equal to the fuselage pitching velocity. Thus, pitching of the tip-path plane lags behind the fuselage pitching motion. The resulting inclination of the tip-path plane relative to the rotor shaft is responsible for the pitch damping of the helicopter. Moreover, the above-mentioned coriolis forces when acting on unbalanced blades cause the blades to twist. Overbalanced blades twist so as to increase the inclination of the tip-path plane and thereby increase the pitch damping. This effect is therefore quite similar to that obtained by increasing the "heaviness" (I_4) of the blade. The opposite is true for underbalanced blades.

The response of the helicopter in pitch to a control step input is shown in Fig. 2 for various values of $\left(\frac{I_2 \Omega^2}{K}\right)$. The response in the first second is shown more accurately in Fig. 3. It is noticed that as the blades become more flexible or more overbalanced, decreased fuselage oscillations of longer period result. This comes about principally through the increased stability in pitch of the craft. However, as pointed out in Section 7, without control displacements in phase with fuselage attitude, the helicopter cannot be stabilized by increased pitch damping. In the limit, only a neutrally stable craft results. The damping rates and frequencies of the unstable fuselage motions of Fig. 2 are as follows:

$\left(\frac{I_2 \Omega^2}{K}\right)$	Damp. factor (m_1)	Frequency (n_1)	Time to double amplitude (sec)	Period (sec)
0	0.188	0.441	3.7	14.2
.2	0.179	0.441	3.9	14.2
1.4	0.070	0.378	10.0	16.6
2.2	0.045	0.344	15.6	18.3

In the case of underbalanced blades, increased fuselage oscillations as well as decreased periods are to be expected due to the decrease in pitch damping.

By the addition of a proper amount of damping to the twisting motions of the blades it is possible to introduce a time lag, and the blades could effectively have a response component in phase with attitude as well as pitching rate. The amount of torsional (structural and aerodynamic) damping introduced by most blades is, however, relatively small.

Blade flexibility and unbalance can materially affect the automatic control system requirements for stability. In fact, consideration of blade flexibility and unbalance could mean the difference between satisfactory or unsatisfactory behavior of a automatic control system. This effect is discussed in Section 7.

b) Aerodynamically Unbalanced Blades

In the case of aerodynamically unbalanced blades the controlling dimensionless parameter was found to be Ch/K, where $Ch = 1/2 \rho a c \Omega^2 R^3 h_1$. "$\rho$" is the mass density of air, "a" the aerodynamic lift coefficient of the blade and "h_1", the aerodynamic chordwise unbalance in feet (positive for forward of elastic axis). The other symbols are as previously defined.

Similar to the action of mass overbalanced blades, aerodynamically underbalanced blades increase the damping in pitch of the helicopter. Fig. 4 shows the relative increase in pitch damping of the helicopter for the case of aerodynamically underbalanced blades. This increase can be attributed to the aeroelastic twisting of the blades. Similar to the action of mass-overbalanced blades, aerodynamically underbalanced blades twist so as to increase the inclination of the rotor tip-path plane relative to the pitching rotor shaft. Aerodynamically overbalanced blades, on the other hand, decrease the inclination and consequently the pitch damping of the helicopter.

The response of the fuselage in pitch to a control step input is shown in Fig. 5 for various values $\left(\frac{C_h}{K}\right)$. The response in the first second is shown more accurately in Fig. 6. As would be expected with increased pitch damping, the amplification of the fuselage oscillation is decreased and the period is increased. The damping factors and frequencies of the unstable oscillations of Fig. 5 are,

$\frac{Ch}{K}$	Damp. factor (m_1)	Frequency(n_1)	Time to double amplitude (sec)	Period (sec)
0	0.188	0.441	3.7	14.2
-2.1	0.136	0.422	5.2	14.9
-3.5	0.106	0.398	6.6	15.8

For the uniform blade considered, the amount of unbalance represented by the above values of Ch/K are,

Ch/K	aerodynamic unbalance (% chord)
0	0
-2.1	5.5%
-3.5	9.1%

As in the case of mass unbalance, aerodynamic unbalance blades do not respond, relative to the rotor shaft, to fuselage attitude but to the rate of pitch of the machine. Consequently, the effect on the automatic control system requirements for stability is similar to that of mass unbalanced blades (Section 7).

4. LIST OF SYMBOLS

a = aerodynamic lift coefficient of the blade.

A = rotor disk area, ft^2.

$a_2 = 1/2 \rho bc \Omega R^2$

b = number of blades

B_1 = control angle of attack input to blades (sine ψ component), radians(Eq.4)

c = blade chord, feet

C_h = blade aerodynamic unbalance parameter, ft. # $= \frac{1}{4} \rho a c \Omega^2 R^3 h_1$

E, L, N, etc = blade motion influence factors (see Eq. 59)

g = acceleration due to gravity, ft/sec^2

h = distance of rotor above helicopter center of gravity(including blades),ft.

h_1 = aerodynamic chordwise unbalance, positive for forward a.c., ft.(Fig.8)

Iy_o = moment of inertia of helicopter about a lateral axis located at helicopter c.g. (including blades), the mass of the blades being considered concentrated at the rotor hub, slug - ft^2

K = blade torsional stiffness (ft.#/radian).

$\overline{m}$ = mass of helicopter, including blades, slugs $= \frac{W}{g}$

R = effective aerodynamic blade radius, ft.

r = radial distance from rotor hub to a blade element, feet (Fig. 7)

t = time, sec.

T = rotor thrust, pounds $= W$

W = gross weight of helicopter, pounds

Ω = rotor angular velocity, radians per sec.

ψ = blade azimuth position, radians (Fig. 7)

β = flapping of blade neutral axis from horizontal plane, radians (Fig. 7)

$= \beta_0 - a_1 \cos\psi - b_1 \sin\psi$

α_1 = fuselage pitch, radians (Fig. 7)

Ξ = chordwise distance of blade center of gravity from neutral axis, positive for forward c.g., ft. (Fig. 8)

$\phi(r)$ = blade torsional mode shape (see Eq. 4).

$\theta(r)$ = geometric angle of attack of a blade airfoil section (Fig. 8)

θ_0 = control collective pitch of the blades

γ_0 = blade twist at tip, collective component, radians (Eq. 4)

γ_1 = blade twist at tip, cosine ψ component, radians (Eq. 4)

γ_2 = blade twist at tip, sine ψ component, radians (Eq. 4)

ρ = mass density of air (slugs/ft^3)

λ_a = dimensionless rotor induced downwash

$$= -\sqrt{\frac{W}{2\rho A \Omega^2 R^2}}$$

γ = dimensionless blade parameter

$$= \frac{\rho a c R^4}{I_1}$$

μ_x = dimensionless horizontal linear velocity of rotor hub

$$= \frac{\dot{x}_0}{\Omega R}$$

$\lambda_{1,2}$ = complex roots of characteristic equation for fuselage oscillation, 1/sec.

$= m_1 \pm i n_1$

Rotor Blade Inertia Integrals:

$$I_1 = \int_{blade} r^2 \, dm \qquad I_2 = \int_{blade} r \xi \, dm$$

$$I_3 = \int_{blade} r \xi \phi(r) \, dm \qquad I_4 = \int_{blade} r \, dm$$

$$I_5 = \int_{blade} \xi \, dm \qquad I_6 = \int_{blade} \xi \phi(r) \, dm$$

$$I_7 = \int_{blade} \xi^2 \, dm \qquad I_8 = \int_{blade} \xi^2 \phi(r) \, dm$$

Rotor Blade Mode Shape Integrals:

$$S_0 = \int_0^R \phi(r) \, dr$$

$$S_1 = \int_0^R r \phi(r) \, dr$$

$$S_2 = \int_0^R r^2 \phi(r) \, dr$$

$$S_3 = \int_0^R r^3 \phi(r) \, dr$$

5. THEORETICAL ANALYSIS OF HELICOPTER HOVERING STABILITY AND CONTROL INCLUDING THE EFFECT OF ROTOR BLADE FLEXIBILITY AND UNBALANCE

Method of Attack

The general method of attack is quite similar to that presented by Nikolsky in Ref. 1, in which however, the blades were assumed to be structurally rigid. In the above reference, the helicopter system was considered to possess three degrees of freedom, namely, fuselage horizontal translation and pitch, and blade flapping. In this analysis to account for blade twist, an additional degree of freedom is introduced. Blade twist is introduced by assuming a general blade torsion mode, ϕ (r). This permits the results of the analysis to be applied to helicopters with blades having any arbitrary torsional stiffness or mass distribution. Basically, four equations of motion are developed to describe the resulting behavior of the helicopter. To preserve the continuity of this development certain parts of Ref. 1 are repeated herein.

Although in the following investigations the motions are assumed to take place in the longitudinal plane of symmetry of the helicopter, the analysis equally holds for the lateral motions. For simplicity, and since this analysis is principally concerned with the effect of blade flexibility and unbalance, the case of a helicopter with no effective flapping hinge offset is treated. However, by use of Ref. 1, and the following procedure, the effect of a flapping hinge offset may be included. In addition, the tip loss factor is accounted for herein, by using an effective aerodynamic rotor radius (R) in all terms except those involving inertia effects, such as the blade inertia integrals (I_1, I_2, etc., pg.12).

Coordinate Axes

The coordinate systems are shown in Fig. 7 and 8. The stationary vertical axis is assumed to coincide with the rotor shaft when the helicopter is in its undisturbed hovering condition. The stationary X_o and Y_o axes are perpendicular to the vertical axis to form a right-handed set of axes.

The moving blade axes (x and y) are taken to coincide, respectively, with the blade's neutral and flapping axes.

Allowing x_b, y_b, and z_b to represent the instantaneous coordinates of a blade element with reference to the stationary axes, the coordinates and their time derivatives are as follows;

$$x_b = x_o - r\cos\psi + \xi\sin\psi$$
$$y_b = -r\sin\psi - \xi\cos\psi$$
$$z_b = \beta r + \xi\theta$$

$$\dot{x}_b = \dot{x}_o + r\Omega\sin\psi + \xi\Omega\cos\psi$$
$$\dot{y}_b = -r\Omega\cos\psi + \xi\Omega\sin\psi$$
$$\dot{z}_b = \dot{\beta} r + \xi\dot{\theta}$$

$$\ddot{x}_b = \ddot{x}_o + r\Omega^2\cos\psi - \xi\Omega^2\sin\psi$$
$$\ddot{y}_b = r\Omega^2\sin\psi + \xi\Omega^2\cos\psi$$
$$\ddot{z}_b = \ddot{\beta} r + \xi\ddot{\theta}$$

where the dots refer to the time derivative of the function.

Blade Flapping Equation

The equation of motion of a blade about its flapping hinge is obtained by equating the total moment at the flapping hinge to zero. Considering moments positive that tend to increase β , this may be stated as,

$$M_y = 0 = (M_y)_m + (M_y)_a \tag{2}$$

where

$(M_y)_m$ = the moment due to the inertia loads.

$(M_y)_a$ = the moment due to the airloads.

The moment due to the inertia loads may be evaluated from the expression,

$$(M_y)_m = -\int_0^R \Big\{ \ddot{z}_b \cos\beta r + \ddot{x}_b \cos\psi (r\beta + \varepsilon\theta) + \ddot{y}_b \sin\psi (r\beta + \varepsilon\theta) \Big\} dm \tag{3}$$

Following the usual convention, the blade flapping angle and blade angle of attack may be expressed as,

$$\beta = \beta_0 - a_1 \cos\psi - b_1 \sin\psi$$
$$\theta(r) = \delta_0 - \delta_1 \cos\psi - \delta_2 \sin\psi \tag{4}$$

where

$$\delta_0 = \theta_0 + \gamma_0 \phi(r)$$
$$\delta_1 = \gamma_1 \phi(r)$$
$$\delta_2 = B_1 + \alpha_1 + \gamma_2 \phi(r)$$

where $\phi(r)$ is the principal torsional deflection mode for the blade, δ_0 is the constant value twist of the blade tip, while δ_1 and δ_2 are, respectively, the cosine and sine components of the blade tip twist. From Eq. (4), the following flapping and blade twisting velocities and accelerations are obtained:

$$\begin{aligned} \dot{\beta} &= -(\dot{a}_1 + b_1\Omega)\cos\psi - (\dot{b}_1 - a_1\Omega)\sin\psi \\ \ddot{\beta} &= -(\ddot{a}_1 + 2\dot{b}_1\Omega - a_1\Omega^2)\cos\psi \\ &\quad - (\ddot{b}_1 - 2\dot{a}_1\Omega - b_1\Omega^2)\sin\psi \\[1em] \dot{\theta}(r) &= -(\dot{\delta}_1 + \delta_2\Omega)\cos\psi - (\dot{\delta}_2 - \delta_1\Omega)\sin\psi \\ \ddot{\theta}(r) &= -(\ddot{\delta}_1 + 2\dot{\delta}_2\Omega - \delta_1\Omega^2)\cos\psi \\ &\quad - (\ddot{\delta}_2 - 2\dot{\delta}_1\Omega - \delta_2\Omega^2)\sin\psi \end{aligned} \tag{5}$$

As β is relatively small for conventional aircraft, $\cos\beta \approx 1$, and the first term on the right hand side of Eq. (3) may be expressed as,

$$\int_0^R \ddot{z}_b\, r\, dm = \int_0^R \ddot{\beta}\, r^2 dm + \int_0^R \varepsilon\, r\, \ddot{\theta}\, dm \tag{6}$$

where

$$\begin{aligned} \int_0^R \ddot{\beta}\, r^2 dm &= -I_1(\ddot{a}_1 + 2\dot{b}_1\Omega - a_1\Omega^2)\cos\psi \\ &\quad - I_1(\ddot{b}_1 - 2\dot{a}_1\Omega - b_1\Omega^2)\sin\psi \end{aligned} \tag{7}$$

$$\int_0^R \xi r \ddot{\theta}\, dm = -\left[I_3 \ddot{\gamma}_1 + 2I_2 \Omega \dot{B}_1 + 2I_2 \Omega \dot{\alpha}_1 + 2I_3 \Omega \dot{\gamma}_2 - I_3 \Omega^2 \gamma_1\right]\cos\psi - \left[I_2 \ddot{B}_1 + I_2 \ddot{\alpha}_1 + I_3 \ddot{\gamma}_2 - 2I_3 \Omega \dot{\gamma}_1 - I_2 \Omega^2 \alpha_1 - I_3 \Omega^2 \gamma_2\right]\sin\psi \qquad (8)$$

The second term on the right hand side of Eq. (3) may be expanded to,

$$\int_0^R \ddot{X}_0 \cos\psi\, r \beta\, dm + \int_0^R \ddot{X}_0 \cos\psi\, \xi \theta\, dm \qquad (9)$$

whose values are,

$$\int_0^R \ddot{X}_0 \cos\psi\, r\beta\, dm = I_1 \Omega^2 \beta_0 \cos^2\psi - I_2 \Omega^2 \beta_0 \sin\psi\cos\psi + I_4 \ddot{X}_0 \beta_0 \cos\psi - \tfrac{3}{4} I_1 \Omega^2 a_1 \cos\psi - \tfrac{1}{4} I_1 \Omega^2 b_1 \sin\psi + \tfrac{1}{4} I_2 \Omega^2 a_1 \sin\psi + \tfrac{1}{4} I_2 \Omega^2 b_1 \cos\psi \qquad (10)$$

$$\int_0^R \ddot{X}_0 \cos\psi\, \xi\theta\, dm = I_2 \Omega^2 \theta_0 \cos^2\psi + I_3 \Omega^2 \gamma_0 \cos^2\psi - I_8 \Omega^2 \gamma_0 \sin\psi\cos\psi + I_5 \ddot{X}_0 \theta_0 \cos\psi + I_6 \ddot{X}_0 \gamma_0 \cos\psi - I_7 \Omega^2 \theta_0 \sin\psi\cos\psi - \tfrac{3}{4} I_3 \Omega^2 \gamma_1 \cos\psi - \tfrac{1}{4} I_2 \Omega^2 (B_1 + \alpha_1)\sin\psi - \tfrac{1}{4} I_3 \Omega^2 \gamma_2 \sin\psi + \tfrac{1}{4} I_8 \Omega^2 \gamma_1 \sin\psi + \tfrac{1}{4} I_7 \Omega^2 (B_1 + \alpha_1)\cos\psi + \tfrac{1}{4} I_8 \Omega^2 \gamma_2 \cos\psi. \qquad (11)$$

Similarly, the final term on the right hand side of Eq. (3) may be expressed as,

$$\int_0^R \ddot{y}_b \sin\psi\, r\beta\, dm + \int_0^R \ddot{y}_b \sin\psi\, \xi\theta\, dm \tag{12}$$

where,

$$\begin{aligned}\int_0^R \ddot{y}_b \sin\psi\, r\beta\, dm = {} & I_1\Omega^2\beta_0 \sin^2\psi + I_2\beta_0\Omega^2\cos\psi\sin\psi \\ & -\tfrac{1}{4}I_1\Omega^2 a_1\cos\psi - \tfrac{3}{4}I_1\Omega^2 b_1\sin\psi \\ & -\tfrac{1}{4}I_2\Omega^2 a_1\sin\psi - \tfrac{1}{4}I_2\Omega^2 b_1\cos\psi\end{aligned} \tag{13}$$

$$\begin{aligned}\int_0^R \ddot{y}_b \xi\theta\sin\psi\, dm = {} & I_2\Omega^2\theta_0\sin^2\psi + I_3\Omega^2\gamma_0\sin^2\psi \\ & + I_7\Omega^2\theta_0\cos\psi\sin\psi + I_8\Omega^2\gamma_0\cos\psi\sin\psi \\ & -\tfrac{1}{4}I_3\Omega^2\gamma_1\cos\psi - \tfrac{3}{4}I_2\Omega^2(B_1+\alpha_1)\sin\psi \\ & -\tfrac{3}{4}I_3\Omega^2\gamma_2\sin\psi - \tfrac{1}{4}I_8\Omega^2\gamma_1\sin\psi \\ & -\tfrac{1}{4}I_7\Omega^2(B_1+\alpha_1)\cos\psi - \tfrac{1}{4}I_8\Omega^2\gamma_2\cos\psi\end{aligned} \tag{14}$$

Thus, the moment due to inertia loads becomes,

$$
\begin{aligned}
\frac{(M_y)_m}{I_1} = {} & \Big[(\ddot{a}_1 + 2\Omega \dot{b}_1) + \frac{I_2}{I_1}(2\Omega \dot{\beta}_1 + 2\Omega \dot{\alpha}_1) \\
& + \frac{I_3}{I_1}(\ddot{\gamma}_1 + 2\Omega \dot{\gamma}_2) - \frac{I_4}{I_1}\ddot{x}_0 \beta_0 - \frac{I_5}{I_1}\ddot{x}_0 \theta_0 \\
& - \frac{I_6}{I_1}\ddot{x}_0 \gamma_0\Big]\cos\psi + \Big[(\ddot{b}_1 - 2\dot{a}_1\Omega) \\
& + \frac{I_2}{I_1}(\ddot{\beta}_1 + \ddot{\alpha}_1) + \frac{I_3}{I_1}(\ddot{\gamma}_2 - 2\Omega\dot{\gamma}_1)\Big]\sin\psi \\
& - \Omega^2\beta_0 - \frac{I_2}{I_1}\Omega^2\theta_0 - \frac{I_3}{I_1}\Omega^2\gamma_0
\end{aligned}
\tag{15}
$$

The moment due to the air loads is

$$
(M_y)_a = \frac{1}{2} a \rho c \int_0^R (\theta U_T^2 + U_P U_T)\, r\, dr \tag{16}
$$

where,

$$
\begin{aligned}
-U_P = {} & \dot{z}_b \cos\beta + (\dot{x}_b \cos\psi + \dot{y}_b \sin\psi)\sin\beta - \lambda_a \Omega R \\
= {} & (r\Omega b_1 + r\dot{a}_1 - \beta_0 \dot{x}_0)\cos\psi + (-r\Omega a_1 + r\dot{b}_1)\sin\psi \\
& + \lambda_a \Omega R
\end{aligned}
$$

$$
\begin{aligned}
U_T = {} & \dot{x}_b \sin\psi - \dot{y}_b \cos\psi \\
= {} & \dot{x}_0 \sin\psi + r\Omega
\end{aligned}
$$

and

$$
\theta(r) = \theta_0 + \gamma_0 \phi(x) - \gamma_1 \phi(r)\cos\psi - [B_1 + \alpha_1 + \gamma_2 \phi(r)]\sin\psi
$$

Consequently,

$$U_T^2 = r^2\Omega^2 + 2\dot{X}_0 r\Omega \sin\psi$$

and

$$\theta U_T^2 = r^2\Omega^2[\theta_0 + \gamma_0\phi(r)] + r^2\Omega^2\gamma_1\phi(r)\cos\psi$$
$$+\{2\dot{X}_0 r\Omega[\theta_0 + \gamma_0\phi(r)] - r^2\Omega^2[B_1 + \alpha_1 + \gamma_2\phi(r)]\}\sin\psi \tag{17}$$

Thus, the first integral of Eq. (16) becomes,

$$\int_0^R \theta U_T^2 r\,dr = \frac{\Omega^2 R^4}{4}\theta_0 + S_3\Omega^2\gamma_0 - S_3\Omega^2\gamma_1\cos\psi$$
$$+\left(\frac{2}{3}\dot{X}_0\Omega R^3\theta_0 + 2S_2\dot{X}_0\Omega\gamma_0 - \Omega^2\frac{R^4}{4}B_1\right.$$
$$\left.-\Omega^2\frac{R^4}{4}\alpha_1 - S_3\Omega^2\gamma_2\right)\sin\psi$$

which, may finally be written as,

$$\frac{\frac{1}{2}\rho ac\int_0^R \theta U_T^2 r\,dr}{I_1} = \frac{\gamma\Omega^2}{8}\left[\theta_0 + \frac{4S_3}{R^4}\gamma_0 - \frac{4S_3}{R^4}\gamma_1\cos\psi\right.$$
$$+\left(\frac{8}{3}\frac{\dot{X}_0}{\Omega R}\theta_0 + 8\frac{S_2}{R^3}\frac{\dot{X}_0}{\Omega R}\gamma_0 - B_1 - \alpha_1\right.$$
$$\left.\left.-4\frac{S_3}{R^4}\gamma_2\right)\sin\psi\right] \tag{18}$$

Evaluation of the second term of Eq. (16) leads to,

$$U_P U_T = \lambda_a \Omega^2 r R + (r^2\Omega^2 b_1 + r^2\Omega \dot{a}_1 - \beta_0 \dot{X}_0 \Omega r)\cos\psi \tag{19}$$
$$+ (-r^2\Omega^2 a_1 + r^2\Omega \dot{b}_1 + \lambda_a \Omega R \dot{X}_0)\sin\psi$$

$$\int_0^R U_P U_T r\,dr = \frac{\lambda_a \Omega^2}{3} + \left(\frac{\Omega^2 R^4 b_1}{4} + \frac{R^4 \Omega \dot{a}_1}{4} \tag{20}$$
$$- \frac{\beta_0 \dot{X}_0 \Omega R^3}{3}\right)\cos\psi + \left(-\frac{R^4}{4}\Omega^2 a_1 + \frac{R^4 \Omega \dot{b}_1}{4}$$
$$+ \lambda_a \frac{\Omega}{2} R^3 \dot{X}_0\right)\sin\psi$$

$$\frac{\frac{1}{2}a\rho c}{I_1}\int_0^R U_P U_T r\,dr = \frac{\gamma\Omega^2}{8}\left[\frac{4}{3}\lambda_a + \left(b_1 + \frac{\dot{a}_1}{\Omega} - \frac{4\beta_0 \dot{X}_0}{3\Omega R}\right)\cos\psi \tag{21}$$
$$+ \left(-a_1 + \frac{\dot{b}_1}{\Omega} + \frac{2\lambda_a \dot{X}_0}{\Omega R}\right)\sin\psi\right]$$

Substituting Eqs. (15) and (16) into Eq. (2) results in an equation whose terms are dependant on sin ψ, cos ψ, or are independent of these harmonic functions. Therefore, in accordance with Fourier's analysis this equation gives rise to three independent equations, namely, constant terms (steady-state),

$$\beta_0 = \frac{\gamma}{8}\left(\theta_0 + \frac{4}{3}\lambda_a + \frac{4S_3}{R^4}\gamma_0\right) - \frac{I_2}{I_1}\theta_0 - \frac{I_3}{I_1}\gamma_0 \tag{22}$$

cosine terms,

$$-\frac{\ddot{X}_0}{\Omega R}\left(\frac{8\beta_0 I_4 R}{I_1 \gamma\Omega} + \frac{8\theta_0 I_5 R}{I_1 \gamma\Omega} + \frac{8\gamma_0 I_6 R}{I_1 \gamma\Omega}\right) - \frac{\dot{X}_0}{\Omega R}\left(\frac{4}{3}\beta_0\right)$$
$$+ \ddot{a}_1\left(\frac{8}{\gamma\Omega^2}\right) + \frac{\dot{a}_1}{\Omega} + \dot{b}_1\left(\frac{16}{\gamma\Omega}\right) + b_1 + \dot{\alpha}_1\left(\frac{I_2}{I_1}\frac{16}{\gamma\Omega}\right)$$
$$+ \left(\frac{8}{\gamma\Omega^2}\frac{I_3}{I_1}\right)\ddot{\gamma}_1 - \gamma_1\left(\frac{4S_3}{R^4}\right) + \dot{\gamma}_2\left(\frac{I_3}{I_1}\frac{16}{\gamma\Omega}\right)$$
$$= -\left(\frac{16}{\gamma\Omega}\frac{I_2}{I_1}\right)\dot{B}_1 \tag{23}$$

sine terms,

$$\frac{\dot{x}_0}{\Omega R}\left[2\left(\frac{4}{3}\theta_0 + \frac{4S_2}{R^3}\gamma_0 + \lambda_a\right)\right] + \ddot{b}_1\left(\frac{8}{\gamma\Omega^2}\right) + \dot{b}_1\left(\frac{1}{\Omega}\right)$$
$$- \dot{a}_1\left(\frac{16}{\gamma\Omega}\right) - a_1 + \ddot{\alpha}_1\left(\frac{I_2}{I_1}\frac{8}{\gamma\Omega^2}\right) - \alpha_1 - \dot{\gamma}_1\left(\frac{I_3}{I_1}\frac{16}{\gamma\Omega}\right) \qquad (24)$$
$$- \alpha_1 - \dot{\gamma}_1\left(\frac{I_2}{I_1}\frac{16}{\gamma\Omega}\right) - \ddot{\gamma}_2\left(\frac{I_3}{I_1}\frac{8}{\gamma\Omega^2}\right) - \frac{4S_2}{R^4}\gamma_2$$
$$= - \ddot{B}_1\left(\frac{I_2}{I_1}\frac{8}{\gamma\Omega^2}\right) + B_1$$

There has, consequently, been obtained three equations to describe the resulting blade flapping motion.

Blade Torsion Equation

The equation of motion of the blade about its neutral axis may be developed by evaluating the moments about this axis due to inertia loads, airloads and elastic restraint. Considering moments positive that tend to decrease the angle of attack, the equation may be stated as,

$$Mx = 0 = (Mx)_m + (Mx)_a + (Mx)_d \qquad (25)$$

where,

$(Mx)_m$ = the moment due to the inertia loads

$(Mx)_a$ = the moment due to the airloads

$(Mx)_d$ = the moment due to the elasticity of the blade.

The moment due to the inertia loads $(M_x)_m$ may be evaluated from the expression:

$$(M_x)_m = \int_0^R [\ddot{z}_b \xi + (\ddot{x}_b \cos\psi + \ddot{y}_b \sin\psi)\beta\xi - \ddot{x}_b \sin\psi\,\xi\theta + \ddot{y}_b \cos\psi\,\xi\theta]\,dm \tag{26}$$

Evaluating the various terms specified above there is obtained,

$$\int_0^R \ddot{z}_b \xi\,dm = \int_0^R \ddot{\beta} r \xi\,dm + \int_0^R \xi^2 \ddot{\theta}\,dm$$

$$= -I_2(\ddot{a}_1 + 2b_1\Omega - a_1\Omega^2)\cos\psi - I_2(\ddot{b}_1 - 2\dot{a}_1\Omega - b_1\Omega^2)\sin\psi - (I_8\ddot{\gamma}_1 + 2I_7\Omega\dot{B}_1 + 2I_7\Omega\dot{\alpha}_1 + 2I_8\Omega\dot{\gamma}_2 - I_8\Omega^2\gamma_1)\cos\psi - (I_7\ddot{B}_1 + I_7\ddot{\alpha}_1 + I_8\ddot{\gamma}_2 - 2I_8\Omega\dot{\gamma}_1 - I_7\Omega^2 B_1 - I_7\Omega^2\alpha_1 - I_8\Omega^2\gamma_2)\sin\psi. \tag{27}$$

$$\int_0^R \ddot{x}_b \cos\psi\,\xi\beta\,dm = I_2\Omega^2\beta_0\cos^2\psi - I_7\Omega^2\beta_0\sin\psi\cos\psi + I_5\ddot{x}_0\beta_0\cos\psi - \frac{3}{4}\Omega^2 I_2 a_1\cos\psi - \frac{1}{4}\Omega^2 I_2 b_1\sin\psi + \frac{1}{4}\Omega^2 I_7 a_1\sin\psi + \frac{1}{4}I_7\Omega^2 b_1\cos\psi \tag{28}$$

$$\int_0^R \ddot{y}_b \sin\psi\,\xi\beta\,dm = I_2\Omega^2\beta_0\sin^2\psi + I_7\Omega^2\beta_0\sin\psi\cos\psi - \frac{1}{4}\Omega^2 I_2 a_1\cos\psi - \frac{3}{4}\Omega^2 I_2 b_1\sin\psi - \frac{1}{4}\Omega^2 I_7 a_1\sin\psi - \frac{1}{4}I_7\Omega^2 b_1\cos\psi \tag{29}$$

$$\int_0^R \ddot{x}_b \sin\psi\, \xi\, \theta\, dm = I_5 \theta_0 \ddot{x}_0 \sin\psi + I_6 \gamma_0 \ddot{x}_0 \sin\psi - I_7 \Omega^2 \theta_0 \sin^2\psi$$
$$- I_8 \Omega^2 \gamma_0 \sin^2\psi - \frac{1}{4} I_3 \Omega^2 \gamma_1 \sin\psi \tag{30}$$
$$+ \frac{1}{4} I_8 \Omega^2 \gamma_1 \cos\psi - \frac{1}{4} I_2 \Omega^2 \beta_1 \cos\psi - \frac{1}{4} I_2 \Omega^2 \alpha_1 \cos\psi$$
$$- \frac{1}{4} I_3 \Omega^2 \gamma_2 \cos\psi + \frac{3}{4} I_7 \Omega^2 \beta_1 \sin\psi$$
$$+ \frac{3}{4} I_7 \Omega^2 \alpha_1 \sin\psi + \frac{3}{4} I_8 \Omega^2 \gamma_2 \sin\psi$$

$$\int_0^R \ddot{y}_b \cos\psi\, \xi\, \theta\, dm = -\frac{1}{4} I_3 \Omega^2 \gamma_1 \sin\psi - \frac{1}{4} I_2 \Omega^2 \beta_1 \cos\psi \tag{31}$$
$$- \frac{1}{4} I_2 \Omega^2 \alpha_1 \cos\psi - \frac{1}{4} I_3 \Omega^2 \gamma_2 \cos\psi + I_7 \Omega^2 \theta_0 \cos^2\psi$$
$$+ I_8 \Omega^2 \gamma_0 \cos^2\psi - \frac{3}{4} I_8 \Omega^2 \gamma_1 \cos\psi - \frac{1}{4} I_7 \Omega^2 \beta_1 \sin\psi$$
$$- \frac{1}{4} I_8 \Omega^2 \gamma_2 \sin\psi$$

Combining Eqs. (27) through (31) as specified in Eq. (26), the moment due to inertia loads becomes equal to,

$$(M_x)_m = I_7 \Omega^2 \theta_0 + I_8 \Omega^2 \gamma_0 + I_2 \Omega^2 \beta_0 + \left[-I_2 (\ddot{a}_1 + 2\dot{b}_1 \Omega) \right.$$
$$- I_8 \ddot{\gamma}_1 - 2 I_7 \Omega \dot{\beta}_1 - 2 I_7 \Omega \dot{\alpha}_1 + 2 I_8 \Omega \dot{\gamma}_2$$
$$\left. + I_5 \ddot{x}_0 \beta_0 \right] \cos\psi + \left[-I_2 (\ddot{b}_1 - 2\dot{a}_1 \Omega) - I_7 \ddot{\beta}_1 \right. \tag{32}$$
$$\left. - I_7 \ddot{\alpha}_1 - I_8 \ddot{\gamma}_2 - I_5 \theta_0 \ddot{x}_0 - I_6 \gamma_0 \ddot{x}_0 + 2 I_8 \Omega \dot{\gamma}_1 \right] \sin\psi$$

The moment due to airloads has the value,

$$(M_x)_a = -\frac{1}{2}\rho a c h_1 \int_0^R \left(U_T^2\theta + U_p U_T\right) dr \tag{33}$$

This may be evaluated simply by use of Eqs. (17) and (19) to yield,

$$\begin{aligned}(M_x)_a = -\frac{1}{2}\rho a c h_1 \Big\{ & \frac{\Omega^2 R^3}{3}\theta_0 + S_2\Omega^2\gamma_0 - S_2\Omega^2\gamma_1\cos\psi \\ & + \Big(\dot{x}_0\Omega R^2\theta_0 + 2S_1\dot{x}_0\Omega\gamma_0 - \frac{\Omega^2R^3}{3}B_1 - \frac{\Omega^2R^3}{3}\alpha_1 \\ & - S_2\Omega^2\gamma_2\Big)\sin\psi + \frac{\lambda_a\Omega^2R^3}{2} + \Big(\frac{\Omega^2R^3}{3}b_1 \\ & + \frac{\Omega R^3}{3}\dot{a}_1 - \frac{\beta_0\Omega R^2}{2}\dot{x}_0\Big)\cos\psi + \Big(-\frac{\Omega^2R^3}{3}a_1 \\ & + \frac{\Omega R^3}{3}\dot{b}_1 + \lambda_a\Omega R^2\dot{x}_0\Big)\sin\psi\Big\}\end{aligned} \tag{34}$$

The moment due to the elasticity of the blade may be written as,

$$(M_x)_d = K\left(\gamma_0 - \gamma_1\cos\psi - \gamma_2\sin\psi\right) \tag{35}$$

where K is the effective torsional stiffness of the blade and root attachment combined.

The blade torsion equation may now be determined by substituting Eqs. (32), (33), and (35) into Eq. (25). Similar to the case of the flapping equation, the resulting torsion equation contains terms which are dependant on sin ψ, cos ψ, or which are independent of these harmonic functions. Therefore, this equation also gives rise to three independent equations, namely, constant terms (steady-state),

$$\begin{aligned}K\gamma_0 + I_7\Omega^2\theta_0 + I_8\Omega^2\gamma_0 + I_2\Omega^2\beta_0 - \frac{1}{2}\rho a c h_1\Big(\frac{\Omega^2R^3}{3}\theta_0 \\ + S_2\Omega^2\gamma_0 + \frac{\lambda_a\Omega^2R^3}{2}\Big) = 0\end{aligned} \tag{36}$$

and since,

$$W = \frac{1}{6}\rho a c b \Omega^2 R^3 \left(\theta_0 + \frac{3}{2}\lambda_a + \frac{3S_2}{R^3}\gamma_0\right) \tag{37}$$

$$K\gamma_0 + I_7\Omega^2\theta_0 + I_8\Omega^2\gamma_0 + I_2\Omega^2\beta_0 - \frac{Wh_1}{b} = 0 \tag{38}$$

cosine terms,

$$\begin{aligned} &-K\gamma_1 - I_2(\ddot{a}_1 + 2\dot{b}_1\Omega) - I_8\ddot{\gamma}_1 - 2I_7\Omega\dot{B}_1 - 2I_7\Omega\dot{\alpha}_1 \\ &+ 2I_8\Omega\dot{\gamma}_2 + I_5\beta_0\ddot{x}_0 - C_h\Big[-\frac{S_2}{R^3}\gamma_1 + \frac{b_1}{3} \\ &+ \frac{\dot{a}_1}{3\Omega} - \frac{\dot{x}_0}{\Omega R}\frac{\beta_0}{2}\Big] = 0 \end{aligned} \tag{39}$$

where $C_h = \frac{1}{2}\rho a c \Omega^2 R^3 h_1$

sine terms,

$$\begin{aligned} &-K\gamma_2 - I_2(\ddot{b}_1 - 2\dot{a}_1\Omega) - I_7\dot{B}_1 - I_7\ddot{\alpha}_1 - I_8\ddot{\gamma}_2 + 2I_8\Omega\dot{\gamma}_1 \\ &-I_5\theta_0\ddot{x}_0 - I_6\gamma_0\ddot{x}_0 - C_h\Big[\frac{\dot{x}_0}{\Omega R}\theta_0 + 2\frac{S_1}{R^2}\frac{\dot{x}_0}{\Omega R}\gamma_0 \\ &-\frac{B_1}{3} - \frac{\alpha_1}{3} - \frac{S_2}{R^3}\gamma_2 - \frac{a_1}{3} + \frac{\dot{b}_1}{3\Omega} + \lambda_a\frac{\dot{x}_0}{\Omega R}\Big] = 0 \end{aligned} \tag{40}$$

Equation of Horizontal Translation

The motion of the helicopter parallel to the x_0 axis, assuming that α_1 is small, may be described by the equation,

$$\bar{m}(\ddot{x}_0 - h\ddot{\alpha}_1) + H_{x_0} = 0 \tag{41}$$

where,

Hx_0 = the component of the left and drag forces, acting on the blades, along the x_0 axis.

$$H_{x_0} = \frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^R \frac{1}{2}\rho bc \left\{ U_T^2 \delta \sin\psi - a(\theta U_T^2 + U_P U_T) \left[(\beta_0 - a_1\cos\psi - b_1\sin\psi)\cos\psi + \frac{U_P}{U_T}\sin\psi\right]\right\} dr \quad (42)$$

In Ref. 1 (pg. 192), the above expression has been evaluated for the case of rigid flapping blades and yields,

$$\begin{aligned}(H_{x_0})_{rigid} = \frac{1}{2}\rho bc\,\Omega R^2 \Big\{ & \frac{\dot{x}_0}{2}\left(\delta + \frac{a\beta_0^2}{2} - a\theta_0\lambda_a\right) + (B_1 + \alpha_1)\left(\frac{a\lambda_a\Omega R}{4}\right) \\ & - \dot{a}_1\left(\frac{a\beta_0}{6}R\right) - \dot{b}_1\left[\frac{aR}{2}\left(\frac{\theta_0}{3} + \lambda_a\right)\right] \\ & + a_1\left[\frac{aR\Omega}{2}\left(\frac{\theta_0}{3} + \frac{\lambda_a}{2}\right)\right] + a_1\left[\frac{a\Omega R}{2}\left(\frac{\theta_0}{3} + \lambda_a\right)\right] \\ & - b_1\left(\frac{a\Omega R\beta_0}{6}\right)\Big\}\end{aligned} \quad (43)$$

However, when considering the flexibility of the blades, this expression must be modified to account for blade twist. Specifically, this necessitates adding to Eq. (42) additional components of the lift, acting on the blades along the x_0 axis, due to blade twist. These components may be evaluated from the expression,

$$(\Delta H_{x_0})_{\substack{blade\\flexibility}} = -\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^R \frac{1}{2}\rho bca(\Delta\theta)U_T^2\left[(\beta_0 - a_1\cos\psi - b_1\sin\psi)\cos\psi + \frac{U_P}{U_T}\sin\psi\right]dr \quad (44)$$

where

$$\Delta\theta = \delta_0\phi(r) - \delta_1\phi(r)\cos\psi - \delta_2\phi(r)\sin\psi$$

Evaluating the various terms of Eq. (44) there results,

$$\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^R \beta_0 \cos\psi (\Delta\theta) U_T^2 \, dr = -\frac{1}{2} S_2 \Omega^2 \beta_0 \gamma_1 \qquad (45)$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^R (-a_1 \cos^2\psi)(\Delta\theta) U_T^2 \, dr = -\frac{1}{2} S_2 \Omega^2 a_1 \gamma_0 \qquad (46)$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\psi \int_0^R (\Delta\theta) U_P U_T \sin\psi \, dr = -\frac{1}{2} S_2 \Omega^2 a_1 \gamma_0 + \frac{1}{2} S_2 \Omega \dot{b}_1 \gamma_0$$

$$+\frac{1}{2} S_0 \lambda_a \Omega R \dot{X}_0 \gamma_0 - \frac{1}{2} S_1 \lambda_a \Omega^2 R \gamma_2 \qquad (47)$$

All other terms equal zero. Consequently,

$$(\Delta Hx_0)_{\substack{\text{blade}\\ \text{flexibility}}} = \frac{1}{2} \rho b c \Omega R^2 \Big[\frac{a}{2} \frac{S_2}{R^2} \Omega \beta_0 \gamma_1 + \frac{a S_2}{R^2} \Omega \gamma_0 a_1$$

$$- \frac{a}{2} \frac{S_2}{R^2} \gamma_0 \dot{b}_1 - \frac{a}{2} S_0 \lambda_a \gamma_0 \frac{\dot{X}_0}{R} + \frac{a}{2} \frac{S_1}{R} \lambda_a \Omega \gamma_2 \Big] \qquad (48)$$

Since,

$$Hx_0 = (Hx_0)_{\text{rigid}} + (\Delta Hx_0)_{\substack{\text{blade}\\ \text{flexibility}}}$$

Adding Eqs. (43) and (48) yields,

$$
\begin{aligned}
H_{x_0} = a_2 \Big\{ & \frac{\dot{x}_0}{2}\left(\delta + \frac{a\beta_0^2}{2} - a\theta_0\lambda_a - a\lambda_a\gamma_0\frac{s_0}{R}\right) + \gamma_1\left(\frac{a\Omega S_2}{2R^2}\beta_0\right) \\
& + (B_1 + \alpha_1)\left(a\frac{\lambda_a\Omega R}{4}\right) + \gamma_2\left(\frac{a\lambda_a\Omega S_1}{2R}\right) \\
& - \dot{a}_1\left(\frac{a\beta_0}{6}R\right) - \dot{b}_1\left[\frac{aR}{2}\left(\frac{\theta_0}{3} + \lambda_a\right)\right] \\
& - \dot{b}_1\left(\frac{aS_2}{2R^2}\gamma_0\right) + a_1\left[a\Omega R\left(\frac{\theta_0}{3} + \frac{3}{4}\lambda_a\right)\right] \\
& + a_1\left[a\frac{S_2}{R^2}\Omega\gamma_0\right] - b_1\left(\frac{a\Omega R}{6}\beta_0\right)\Big\}
\end{aligned} \tag{49}
$$

where $a_2 = \frac{1}{2}\rho bc\Omega R^2$

The equation for horizontal translation becomes,

$$
\begin{aligned}
& \dot{\mu}_x H_{\dot{\mu}_x} + \mu_x H_{\mu_x} + \ddot{\alpha}_1 H_{\ddot{\alpha}_1} + \alpha_1 H_{\alpha_1} + \dot{a}_1 H_{\dot{a}_1} \\
& + a_1 H_{a_1} + \dot{b}_1 H_{\dot{b}_1} + b_1 H_{b_1} + \gamma_1 H_{\gamma_1} + \gamma_2 H_{\gamma_2} = + B_1 H_{B_1}
\end{aligned} \tag{50}
$$

where,

$$
\begin{aligned}
H_{\dot{\mu}_x} &= \Omega R\overline{m} \\
H_{\mu_x} &= \frac{a_2\Omega R}{2}\left(\delta + \frac{a\beta_0^2}{2} - a\theta_0\lambda_a - a\lambda_a\frac{S_0}{R}\gamma_0\right) \\
H_{\ddot{\alpha}_1} &- \overline{m}h \\
H_{\alpha_1} &= \frac{aa_2\Omega R\lambda_a}{4} \\
H_{\dot{a}_1} &= -\frac{aa_2\beta_0 R}{6} \\
H_{a_1} &= aa_2\Omega R\left(\frac{\theta_0}{3} + \frac{3}{4}\lambda_a\right) + \frac{aa_2S_2\Omega\gamma_0}{R^2} \\
&= aa_2\frac{\Omega R\lambda_a}{4} + T \\
H_{b_1} &= -aa_2\frac{\Omega R\beta_0}{6}
\end{aligned} \tag{50a}
$$

$$H_{\dot{b}_1} = -\frac{a a_2 R}{2}\left(\frac{\theta_0}{3} + \lambda_a\right) - \frac{a a_2 S_2}{2R^2}\gamma_0$$

$$H_{\gamma_1} = \frac{a a_2 \Omega S_2 \beta_0}{2R^2}$$

$$H_{\gamma_2} = \frac{a a_2 \lambda_a \Omega S_1}{2R}$$

$$H_{B_1} = -\frac{a a_2 \lambda_a \Omega R}{4}$$

Equation of Fuselage Pitch

The motion of the helicopter in pitch about the y_0 axis may be described as,

$$I_{y_0}\ddot{\alpha}_1 + H_{x_0} h + T h \alpha_1 = 0 \tag{51}$$

By use of Eq. (49), the above equation may be written in the form,

$$\begin{aligned} &\mu_x M_{y_0\mu_x} + \ddot{\alpha}_1 M_{y_0\ddot{\alpha}_1} + \alpha_1 M_{y_0\alpha_1} + \dot{a}_1 M_{y_0\dot{a}_1} + a_1 M_{y_0 a_1} \\ &\quad + \dot{b}_1 M_{y_0\dot{b}_1} + b_1 M_{y_0 b_1} + \gamma_1 M_{y_0\gamma_1} + \gamma_2 M_{y_0\gamma_2} = + B_1 M_{y_0 B_1} \end{aligned} \tag{52}$$

where

$$\begin{aligned} M_{y_0\mu_x} &= h H_{\mu_x} \\ M_{y_0\ddot{\alpha}_1} &= I_{y_0} \\ M_{y_0\alpha_1} &= Th + h H_{\alpha_1} \\ M_{y_0\dot{a}_1} &= h H_{\dot{a}_1} \\ M_{y_0 a_1} &= h H_{a_1} \\ M_{y_0\dot{b}_1} &= h H_{\dot{b}_1} \\ M_{y_0 b_1} &= h H_{b_1} \\ M_{y_0\gamma_1} &= h H_{\gamma_1} \\ M_{y_0\gamma_2} &= h H_{\gamma_2} \\ M_{y_0 B_1} &= h H_{B_1} \end{aligned} \tag{52a}$$

The motion of the disturbed helicopter, under the assumptions made, have now been fully specified by eight equations, two steady-state and six disturbed equations of motion. The latter consist of two blade flapping equations, two blade torsion equations, the equation of horizontal translation and the equation of fuselage pitch. However it is immediately obvious that the solution of the six equations would be extremely tedious and time consuming. It would therefore be desirable to simplify these equations, if possible, without any great loss in accuracy. Such a simplification can be accomplished following the procedure presented in Ref. 2.

Simplified Equations of Motion

The quantitative determination of the response of a helicopter may be greatly facilitated by simplifying the blade equations.

In Ref. 2, it has been shown that the flapping blade motions consist of high-frequency heavily damped oscillations. The order of magnitude of these oscillations was such as to suggest that their influence on the motion of the helicopter would be small, i.e., the blade response following a disturbance is rapid (within a couple of rotor revolutions) compared to the long-period fuselage oscillation which is of principal concern. In fact, the disturbance of the flapping motion is reduced to a tenth of its initial value in $\frac{40}{2\pi\gamma}$ revolutions of the blade. Based on this, little error appears to be introduced by neglecting all acceleration terms proportional to $\ddot{a}_1$ and $\ddot{b}_1$. It is also realized that $\dot{a}_1$ is approximately equal to $\dot{\alpha}$, since a_1 approximately equals α_1 , and both quantities oscillate at the same frequency. However, since small differences in a_1 and α_1 introduce major contributions to the response of a helicopter it cannot be assumed that a_1 equals α_1 . The coefficients of $\dot{b}_1$ appear small and since $\dot{b}_1$ itself would be small, its contribution may be neglected.

From the characteristics of typical rotor blades it may be established that, $I_5 << I_1$; $I_6 << I_1$; $I_2 << I_1$; $I_3 << I_1$; $I_7 << I_2$; and $I_8 << I_2$.

Accepting the above simplifications appears not to affect the long-period oscillations to any marked degree. However, under these simplifications the equations of motion of the blades about the flapping hinge will permit direct solution for a_1 and b_1 in terms of the other degrees of freedom.

By extension of the above arguments to the case of the blade torsion equations, the contribution of the dynamic terms $\ddot{\gamma}_1, \dot{\gamma}_1, \ddot{\gamma}_2$ and $\dot{\gamma}_2$ may be recognized as small. Thus, the simplified blade flapping and torsion equations are as follows.

Simplified Blade Flapping Equations

constant term (steady-state),

$$\beta_0 = \frac{\gamma}{8}\left(\theta_0 + \frac{4}{3}\lambda_a + \frac{4S_3}{R^4}\gamma_0\right) \tag{53}$$

cosine term,

$$b_1 = -\dot{u}_x M_{y\dot{u}_x} - u_x M_{yu_x} + \dot{\alpha}_1 M_{y\dot{\alpha}_1} - \gamma_1 M_{y\gamma_1} \tag{54}$$

where

$$M_{y\dot{\mu}x} = -\frac{8B_0 I_4 R}{I_1 \gamma \Omega}$$

$$M_{y\mu x} = -\frac{4}{3}\beta_0 \qquad (54a)$$

$$M_{y\dot{a}_1} = \frac{1}{\Omega}$$

$$M_{y\gamma_1} = -\frac{4S_3}{R^4}$$

sine term,

$$a_1 = \mu_x M'_{y\mu x} - \alpha_1 - \dot{\alpha}_1 M'_{y\dot{a}_1} + \gamma_2 M'_{y\gamma_2} - B_1 \qquad (55)$$

where

$$M'_{y\mu x} = 2\left(\frac{4}{3}\theta_0 + 4\frac{S_2}{R^3}\gamma_0 + \lambda_a\right)$$

$$M'_{y\dot{a}_1} = -\frac{16}{\gamma\Omega}$$

$$M'_{y\gamma_2} = -\frac{4S_3}{R^4} \qquad (55a)$$

Simplified Blade Torsion Equations

constant term (steady-state),

$$\gamma_0 = \frac{Wh_1}{Kb} - \frac{I_2\Omega^2}{K}\beta_0 \qquad (56)$$

cosine term,

$$\gamma_1 = -\mu_x\frac{T_{\mu x}}{T_{\gamma_1}} + \dot{\alpha}_1\frac{T_{\dot{a}_1}}{T_{\gamma_1}} + b_1\frac{T_{b_1}}{T_{\gamma_1}} \qquad (57)$$

where,

$$T_{\mu x} = \frac{C_h\beta_0}{2}$$

$$T_{\dot{a}_1} = -\frac{C_h}{3\Omega}$$

$$T_{b_1} = \frac{C_h}{3} \qquad (57a)$$

$$T_{\gamma_1} = -\left(K - \frac{C_h S_2}{R^3}\right)$$

sine term

$$\gamma_2 = -\mu_x \frac{T'_{\mu x}}{T'_{\gamma_2}} - \alpha_1 \frac{T'_{\alpha_1}}{T'_{\gamma_2}} + \dot{\alpha}_1 \frac{T'_{\dot{a}_1}}{T'_{\gamma_2}} - a_1 \frac{T'_{a_1}}{T'_{\gamma_2}} + B_1 \frac{T'_{B_1}}{T'_{\gamma_2}} \tag{58}$$

where

$$T'_{\mu x} = -C_h\left(\theta_0 + \lambda_a + \frac{2S_1}{R^2}\gamma_0\right)$$

$$T'_{\alpha_1} = \frac{C_h}{3}$$

$$T'_{\dot{a}_1} = 2 I_2 \Omega$$

$$T'_{a_1} = \frac{C_h}{3} \tag{58a}$$

$$T'_{\gamma_2} = -\left(K - \frac{C_h S_2}{R^3}\right)$$

$$T'_{B_1} = -\frac{C_h}{3}$$

It is now possible to solve for the quantities a_1, b_1, γ_1 and γ_2 in terms of the two degrees of freedom, fuselage translation, and pitch. This results in the following values,

$$a_1 = \mu_x D - \alpha_1 - \dot{\alpha}_1 F - B_1$$

$$b_1 = -\dot{\mu}_x A - \mu_x H + \dot{\alpha}_1 J \tag{59}$$

$$\gamma_1 = -\dot{\mu}_x C - \mu_x E$$

$$\gamma_2 = -\mu_x M + \dot{\alpha}_1 P$$

where, letting $L = 1 + \frac{T_{a_1}}{T'_{\gamma_2}} M'_{y\gamma_2}$,

$$D = \left(M'_{y\mu x} - \frac{T'_{\mu x}}{T_{\gamma_2}} M'_{y\gamma_2}\right) \Big/ L$$

$$F = \left(M'_{y\dot{a}_1} - \frac{T'_{\dot{a}_1}}{T'_{\gamma_2}} M'_{y\gamma_2}\right) \Big/ L$$

$$M = \frac{\left(\frac{T'_{\mu x}}{T'_{\gamma_2}} + M'_{y\mu x}\frac{T'_{a_1}}{T'_{\gamma_2}}\right)}{L}$$

$$P = \frac{\left(\frac{T'_{\dot{a}_1}}{T'_{\gamma_2}} + M'_{y\dot{a}_1}\frac{T'_{a_1}}{T'_{\gamma_2}}\right)}{L}$$

and, letting $N = 1 + \frac{T_{b_1}}{T_{\gamma_1}} M_{y\gamma_1}$

$$A = \frac{M_{y\dot{\mu}x}}{N}$$

$$H = \frac{\left(M_{y\mu x} - \frac{T_{\mu x}}{T_{\gamma_1}} M_{y\gamma_1}\right)}{N}$$

$$J = \frac{\left(M_{y\dot{a}_1} - \frac{T_{\dot{a}_1}}{T_{\gamma_1}} M_{y\gamma_1}\right)}{N}$$

$$C = \frac{\frac{T_{b_1}}{T_{\gamma_1}} M_{y\dot{\mu}x}}{N}$$

$$E = \frac{\left(\frac{T_{\mu x}}{T_{\gamma_1}} + \frac{T_{b_1}}{T_{\gamma_1}} M_{y\mu x}\right)}{N}$$

Substituting Eqs. (59) into Eqs. (50) and (52), the simplified equations of fuselage translation and pitch are thereby obtained.

Simplified Equation of Horizontal Translation

$$\dot{\mu}_x \bar{H}_{\dot{\mu}x} + \mu_x \bar{H}_{\mu x} + \ddot{\alpha}_1 \bar{H}_{\ddot{\alpha}_1} + \dot{\alpha}_1 \bar{H}_{\dot{\alpha}_1} + \alpha_1 \bar{H}_{\alpha_1} = B_1 \bar{H}_{B_1} \qquad (60)$$

where

$$\bar{H}_{\dot{\mu}x} = H_{\dot{\mu}x} - H_{b_1} A - H_{\gamma_1} C$$

$$\bar{H}_{\mu x} = H_{\mu x} + H_{a_1} D - H_{b_1} H - H_{\gamma_1} E - H_{\gamma_2} M$$

$$\bar{H}_{\ddot{\alpha}_1} = H_{\ddot{\alpha}_1}$$

$$\bar{H}_{\dot{\alpha}_1} = -H_{\dot{a}_1} - H_{a_1} F + H_{b_1} J + H_{\gamma_2} P \qquad (60a)$$

$$\bar{H}_{\alpha_1} = H_{\alpha_1} - H_{a_1} = -T$$

$$\bar{H}_{B_1} = H_{B_1} + H_{a_1} = T$$

Simplified Equation of Fuselage Pitch

$$\dot{\mu}_x \overline{M_{y_0 \dot{\mu}x}} + \mu_x \overline{M_{y_0 \mu x}} + \ddot{\alpha}_1 \overline{M_{y_0 \ddot{\alpha}_1}} + \dot{\alpha}_1 \overline{M_{y_0 \dot{\alpha}_1}} = B_1 \overline{M_{y_0 B_1}} \qquad (61)$$

where

$$\overline{M_{y_0 \dot{\mu}_x}} = -M_{y_0 b_1} A - M_{y_0 \gamma_1} C$$

$$\overline{M_{y_0 \mu_x}} = h \overline{H_{\mu_x}}$$

$$\overline{M_{y_0 \ddot{\alpha}_1}} = M_{y_0 \ddot{\alpha}_1}$$

$$\overline{M_{y_0 \dot{\alpha}_1}} = h \overline{H_{\dot{\alpha}_1}} \qquad (61a)$$

$$\overline{M_{y_0 B_1}} = h \overline{H_{B_1}}$$

Response of a Helicopter to a Control Step Input

The response of a helicopter to a control input may now be easily determined by use of Eqs. (60) and (61) and Operational Calculus. The general procedure for the determination of the fuselage pitch response is briefly discussed below. For the response in translation (μ_x), and a detailed description of the use of operational calculus for stability analyses, use of Ref. 1 is suggested.

Taking the Laplace transform of the above two equations there is obtained,

$$\mu_x(\lambda) A_{11} + \alpha_1(\lambda) A_{12} = A_{13} B_1$$
$$\mu_x(\lambda) A_{21} + \alpha_1(\lambda) A_{22} = A_{23} B_1 \qquad (62)$$

where,

$$A_{11} = \overline{H_{\dot{\mu}x}}\lambda + \overline{H_{\mu x}}$$

$$A_{12} = \overline{H_{\ddot{\alpha}_1}}\lambda^2 + H_{\dot{\alpha}_1}\lambda + \overline{H_{\alpha_1}}$$

$$A_{21} = M_{yo\dot{\mu}x}\lambda + \overline{M_{yo\mu x}}$$

$$A_{22} = M_{yo\ddot{\alpha}_1}\lambda^2 + \overline{M_{yo\dot{\alpha}_1}}\lambda$$

$$A_{13} = \overline{H_{B_1}}$$

$$A_{23} = h\overline{H_{B_1}}$$

The transfer functions is given by,

$$\frac{\alpha_1(\lambda)}{B_1} = \frac{\begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix}}{\Delta} \tag{63}$$

in which

$$\Delta = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \tag{64}$$

$$= \overline{b_3}\lambda^3 + \overline{b_2}\lambda^2 + \overline{b_1}\lambda + \overline{b_0}$$

where

$$\overline{b_3} = \overline{H_{\dot{\mu}x}}\,\overline{M_{yo\ddot{\alpha}_1}} - \overline{H_{\ddot{\alpha}_1}}\,M_{yo\dot{\mu}x}$$

$$\overline{b_2} = \overline{H_{\dot{\mu}x}}\,\overline{M_{yo\dot{\alpha}_1}} + \overline{H_{\mu x}}\,\overline{M_{yo\ddot{\alpha}_1}} - \overline{H_{\ddot{\alpha}_1}}\,\overline{M_{yo\mu x}} - \overline{H_{\dot{\alpha}_1}}\,\overline{M_{yo\dot{\mu}x}} \tag{64a}$$

$$\overline{b_1} = -\overline{H_{\alpha_1}}\,\overline{M_{yo\dot{\mu}x}} = T\,\overline{M_{yo\dot{\mu}x}}$$

$$\overline{b_0} = -\overline{H_{\alpha_1}}\,\overline{M_{yo\mu x}} = T\,\overline{M_{yo\mu x}}$$

According to Routh's stability criterion, dynamic stability only occurs if the coefficients $\overline{b}_3$, $\overline{b}_2$, $\overline{b}_1$, and $\overline{b}_o$ are positive and if,

$$\overline{b}_2 \overline{b}_1 > \overline{b}_3 \overline{b}_o \tag{64b}$$

For a typical helicopter, $\overline{b}_1$ is negative, and therefore is inherently dynamically unstable. By inspection of the terms involved in the above equations, it can be seen that the major effect of blade flexibility and unbalance is shown in the factors $\overline{My_{o\dot{\alpha}_1}}$ and $\overline{H_{\dot{\alpha}_1}}$, and consequently $\overline{b}_2$. The values of $\overline{b}_3$, $\overline{b}_2$ and $\overline{b}_o$ remain essentially unaffected. For mass-overbalanced or aerodynamically under-balanced blades $\overline{My_{o\dot{\alpha}_1}}$ and $\overline{H_{\dot{\alpha}_1}}$ are increased and vice versa. This is, therefore, equivalent to increasing or decreasing the "heaviness" (coriolis effect) of the blades, which is associated with damping in pitch of the rotor.

Expanding Eq. (63) there results

$$\frac{\alpha_1(\lambda)}{B_1} = \frac{d_1 \lambda}{b_3 \lambda^3 + b_2 \lambda^2 + b_1 \lambda + b_o} \tag{65}$$

where

$$d_1 = \left(h \overline{H_{\dot{\mu}x}} - \overline{M_{yo\dot{\mu}x}}\right) \overline{H_{B_1}} \tag{65a}$$

Setting $\Delta = 0$ yields, for conventional type helicopters, one real root and one complex pair, say

$$\begin{aligned} \lambda_{1,2} &= m_1 \pm i n_1 \\ \lambda_3 &= g_1 \end{aligned} \tag{66}$$

Thus, the time history of the pitching motion is,

$$\frac{\alpha_1(t)}{B_1} = K_1 e^{g_1 t} + K_2 e^{m_1 t} \cos n_1 t + K_3 e^{m_1 t} \sin n_1 t \tag{67}$$

where $K_1 = \dfrac{d_1}{3\bar{b}_3 g_1^2 + 2\bar{b}_2 g_1 + \bar{b}_1}$

By use of the boundary conditions, $\alpha_1 = \dot{\alpha}_1 = 0$ at t = 0, the constants K_2 and K_3 may be evaluated and thereby $\frac{\alpha_1(t)}{B_1}$. A similar approach may be used to determine $\frac{u_x(t)}{B_1}$. However, since the characteristics of the motion can be seen by use of either $\frac{\alpha_1(t)}{B_1}$ or $\frac{u_x(t)}{B_1}$ there is no need to present, herein, the solution for $\frac{u_x(t)}{B_1}$.

6. Procedure and Application of Method

To demonstrate the procedure and application of the method derived in Section 5, a typical helicopter, equipped with aerodynamically underbalanced blades, will be analyzed. The aircraft considered is the same as that analyzed in Ref. 1, and has the following specifications,

$W = 5000\#$

$R = 24$ ft.(the aerodynamic effective radius)

$b = 3$

$c = 1.5$ft. (the effective chord)

$\Omega = 20.3$ rad/sec

$I_1 = 540$ slug ft^2

$I_2 = 0$

$W_b = 102.2\#$ (the weight of each blade)

$Iy_o = 5775$ slug ft^2 (including blade mass at hub)

$a = 5.75$ per radian

$\rho = .00238$ slugs-ft^3

$h = 6.25$ ft.

$\delta = 0.018$

$$\gamma = \frac{c\rho a R^4}{I_1} = 12.65$$

$$\lambda_a = -\sqrt{\frac{W}{2\rho A \Omega^2 R^2}} = -0.0495$$

$a_2 = 62.7$

$h_1 = -0.082$ ft

$K = 2300$ ft.#/rad

$C_h/K = -2.1$

The step-by-step procedure is as follows:

a) Based on the blade's mass and structural characteristics determine $\phi(r)$. This may be approximated by the first torsional mode shape of the blade. For a uniform blade, $\phi(r) = \sin \frac{\pi r}{2R}$.

b) Evaluate the mode shape integrals (pg. 12)

$$S_0 = 0.637R$$
$$S_1 = 0.405R^2$$
$$S_2 = 0.296R^3$$
$$S_3 = 0.231R^4$$

c) Determine the steady state values of θ_0, β_0, and γ_0 from the equations,

$$\beta_0 = \frac{\gamma}{8}\left(\theta_0 + \frac{4}{3}\lambda_a + \frac{4S_3}{R^4}\gamma_0\right) \tag{53}$$

$$W = \frac{1}{6}\rho a c b \Omega^2 R^3\left(\theta_0 + \frac{3}{2}\lambda_a + \frac{3S_2}{R^3}\gamma_0\right) \tag{37}$$

$$\gamma_0 = \frac{W}{a a_2 \Omega R}\left(\frac{C_h}{K}\right) - \frac{I_2 \Omega^2}{K}\beta_0 \tag{56}$$

thus,

$$\beta_0 = 0.145$$
$$\theta_0 = 0.213$$
$$\gamma_0 = -0.06$$

d) Evaluate the stability derivatives Eqs (50a) (52a), (54a), (55a), (57a), and (58a).

$F_{\dot{\mu}x} = 75{,}600$

$H_{\mu x} = 1958$

$H_{\ddot{\alpha}_1} = -970$

$H_{\alpha_1} = -2175$

$H_{\dot{a}_1} = -209.1$

$H_{a_1} = 2825$

$H_{\dot{b}_1} = -16.3$

$H_{b_1} = -4245$

$H_{\gamma_1} = 3769$

$H_{\gamma_2} = -1761$

$H_{B_1} = 2175$

$My_{o_{\mu x}} = 12{,}240$

$My_{o\ddot{\alpha}_1} = 5775$

$My_{o\alpha_1} = 17{,}656$

$My_{o\dot{a}_1} = -1307$

$My_{oa_1} = 17{,}656$

$My_{o\dot{b}_1} = -101.9$

$My_{o\,b_1} = -26{,}530$

$My_{o\gamma_1} = 23{,}559$

$My_{o\gamma_2} = -11{,}004$

$My_{oB_1} = -13{,}594$

$T\mu_x = -635.3$

$T\dot{a}_1 = 143.9$

$Tb_1 = -2921$

$T\gamma_1 = -6761$

$T'\mu_x = 1008$

$T'_{\alpha_1} = -2921$

$T'\dot{a}_1 = 0$

$T'a_1 = -2921$

$T'\gamma_2 = -6761$

$T'_{B_1} = 2921$

$My_{\dot{\mu}_x} = -0.00765$

$My_{\mu_x} = -0.1933$

$My_{\dot{a}_1} = 0.0493$

$My_{\gamma_1} = -0.924$

$My'_{\mu x} = 0.3272$

$My'_{\dot{a}_1} = -0.0623$

$My'_{\gamma_2} = -0.924$

e) Determine blade motion influence factors Eq. (59).

$L = 0.6008$

$D = 0.3154$

$F = -0.1037$

$M = -0.0128$

$P = -0.0449$

$N = 0.6008$

$A = -0.0127$

$H = -0.1772$

$J = 0.0493$

$C = -0.0055$

$E = 0.0174$

f) Evaluate stability derivatives Eqs (60a) and (61a)

$\overline{H_{\dot{\mu}_x}} = 75{,}567$

$\overline{H_{\mu_x}} = 2{,}009$

$\overline{H_{\ddot{\alpha}_1}} = -970$

$\overline{H_{\dot{\alpha}_1}} = 371.9$

$\overline{H_{\alpha_1}} = -5000$

$\overline{H_{B_1}} = 5000$

$\overline{My_{o\dot{\mu}_x}} = -208.2$

$\overline{My_{o\mu_x}} = 12{,}554$

$\overline{My_{o\ddot{\alpha}_1}} = 5775$

$\overline{My_{o\dot{\alpha}_1}} = 2324$

$\overline{My_{oB_1}} = 31{,}250$

g) Determine characteristic equation coefficients Eq. (64)

$\overline{b}_3 = 436.2 \times 10^6$

$\overline{b}_2 = 199.5 \times 10^6$

$\overline{b}_1 = -1.041 \times 10^6$

$\overline{b}_o = 62.77 \times 10^6$

h) Determine roots of characteristic equation, $\Delta = 0$.

$$\lambda_{1,2} = 0.136 \pm 0.422\, i$$

$$\lambda_3 = -0.730$$

i) Evaluate K_1, K_2, and K_3, Eq. (67)

$$K_1 = 5.833$$

$$K_2 = -5.833$$

$$K_3 = 11.969$$

j) Thus, the response is,

$$\frac{\alpha_1(t)}{B_1} = 5.833e^{-0.730t} - 5.833e^{0.136t} \cos 0.422t$$

$$+ 11.969e^{0.136t} \sin 0.422t.$$

A graph of this response is shown in Fig. 5.

7. Effect of Blade Flexibility on Automatic Control System Requirements

In general, for the rapid subsidence of a disturbance of a helicopter, periodic control displacements (which tilt the rotor tip-path plane relative to the rotor shaft) in phase with fuselage attitude (α_1) and pitching velocity (damping) are required (Ref. 6). The first corresponds to a kind of static stability and decreases the period of the motion. The latter increases the period, and when applied alone tends to prevent a further increase of the disturbance. Moreover, the latter has a remarkable stabilizing effect when combined with the first. By proper combination of the two any helicopter can be stabilized.

Since it has been shown that blade flexibility and unbalance noticeably affects the damping in pitch of the helicopter, it is reasonable to expect that these factors would also affect the automatic control system requirements for stability. To demonstrate this fact, consider a helicopter equipped with an attitude gyro system so as to yield control displacements in phase with the fuselage attitude $(-B_A\alpha_1)$. For this case, Eqs. (60) and (61) are expanded to,

$$\dot{\mu}_x \overline{H_{\dot{\mu}_x}} + \mu_x \overline{H_{\mu_x}} + \ddot{\alpha}_1 \overline{H_{\ddot{\alpha}_1}} + \dot{\alpha}_1 \overline{H_{\dot{\alpha}_1}} - T(1-B_A)\alpha_1 = TB_1 \qquad (68)$$

$$\dot{\mu}_x \overline{M_{y_0\dot{\mu}_x}} + \mu_x \overline{M_{y_0\mu_x}} + \ddot{\alpha}_1 \overline{M_{y_0\ddot{\alpha}_1}} + \dot{\alpha}_1 \overline{M_{y_0\dot{\alpha}_1}} + ThB_A\alpha_1 = ThB_1 \qquad (69)$$

The coefficients of the characteristic equation become,

$$\bar{b}_3 = \overline{H_{\dot{\mu}x}}\;\overline{M_{yo\dot{\alpha}_1}} - \overline{H_{\dot{\alpha}_1}}\;\overline{M_{yo\dot{\mu}x}}$$

$$\bar{b}_2 = \overline{H_{\dot{\mu}x}}\;\overline{M_{yo\alpha_1}} + \overline{H_{\mu x}}\;\overline{M_{yo\dot{\alpha}_1}} - \overline{H_{\dot{\alpha}_1}}\;\overline{M_{yo\mu x}} - \overline{H_{\alpha_1}}\;\overline{M_{yo\dot{\mu}x}} \qquad (70)$$

$$\bar{b}_1 = Th\,\overline{H_{\dot{\mu}x}}\,B_A + T(1-B_A)\,\overline{M_{yo\dot{\mu}x}}$$

$$\bar{b}_0 = T\,\overline{M_{yo\mu x}}$$

It is noticed that while pitch damping ($M_{yo\dot{\alpha}_1}$) affects the coefficient $\bar{b}_2$, automatic attitude control (B_A) affects the coefficient $\bar{b}_1$. The stability condition Eq. (64b) may thereby be satisfied. It is interesting to notice that without attitude control ($B_A = 0$), stability cannot be obtained. In other words, increasing the pitch damping of a helicopter can only lead to a neutrally stable craft.

To quantitatively show how flexibility and unbalance affect the automatic control system requirements, assume a system in which $B_A = 0.09$. The coefficients (Eq. 70) for various values of mass unbalanced blades become,

$\frac{I_2\Omega^2}{K}$	$\bar{b}_3$	$\bar{b}_2$	$\bar{b}_1$	$\bar{b}_o$
0	436×10^6	106×10^6	218×10^6	62.3×10^6
1.4	436×10^6	393×10^6	218×10^6	68.6×10^6
2.2	436×10^6	558×10^6	218×10^6	71.7×10^6

The frequencies and damping factors of the fuselage oscillations are therefore,

$\frac{I_2 \Omega^2}{K}$	Damp. factor (m_1)	Frequency (n_1)	Time to half amplitude (sec)	Period (sec)
0	0	0.705	∞	8.9
1.4	-.175	0.500	4	12.5
2.2	-.167	0.420	4.2	15.0

Thus, a neutrally stable machine of short period results when balanced or rigid blades are assumed. The machine becomes more stable as the degree of overbalance or flexibility increases. Finally a very satisfactory stable oscillation of long period is obtained. In the case of underbalanced blades, unstable oscillations would result. Thus, consideration of blade flexibility and unbalance can often mean the difference between a satisfactory or unsatisfactory automatic control system design.

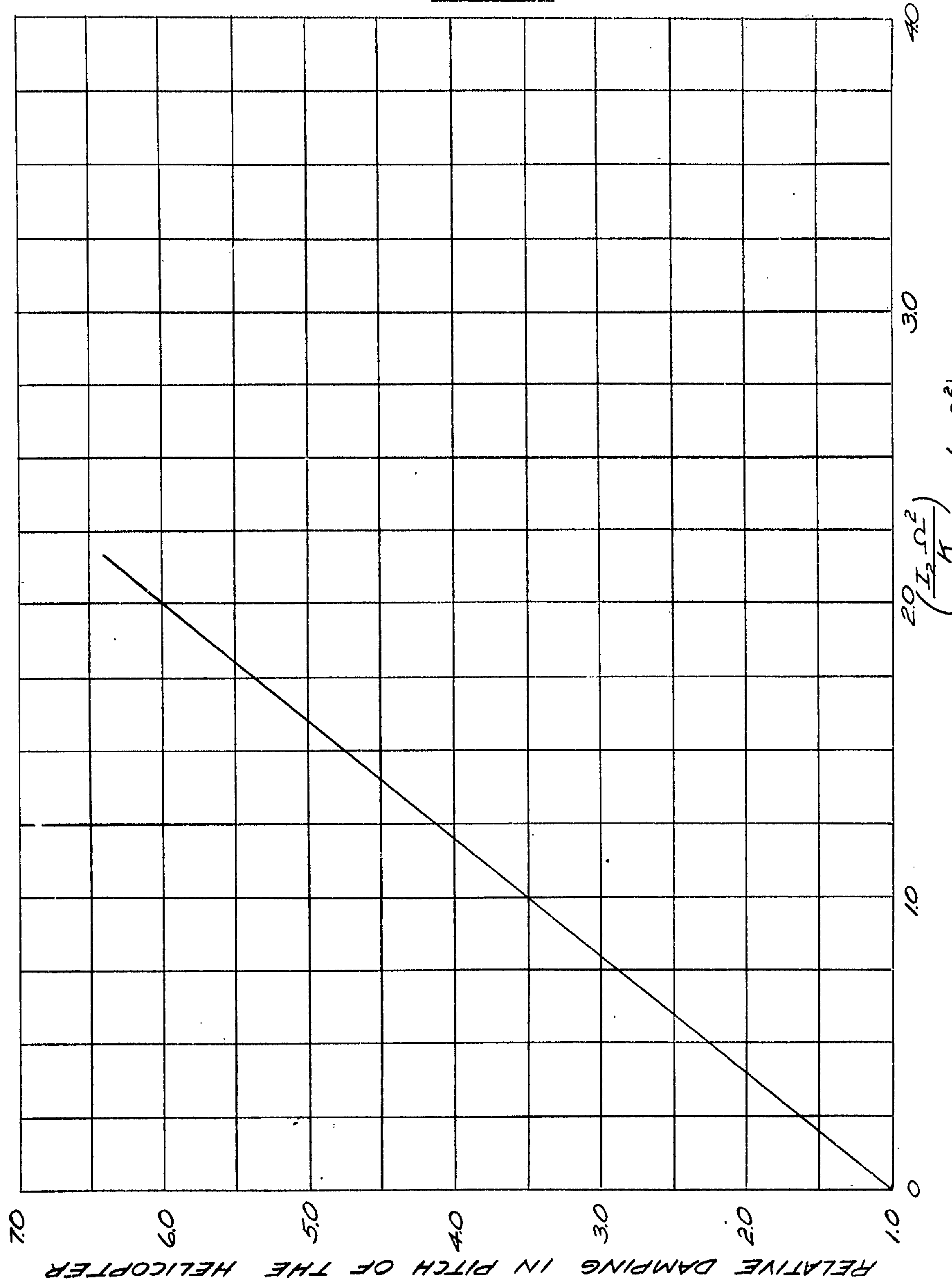

FIG. 1 Relative Damping in Pitch of the Helicopter vs. $\left(\frac{I_2 \Omega^2}{K}\right)$

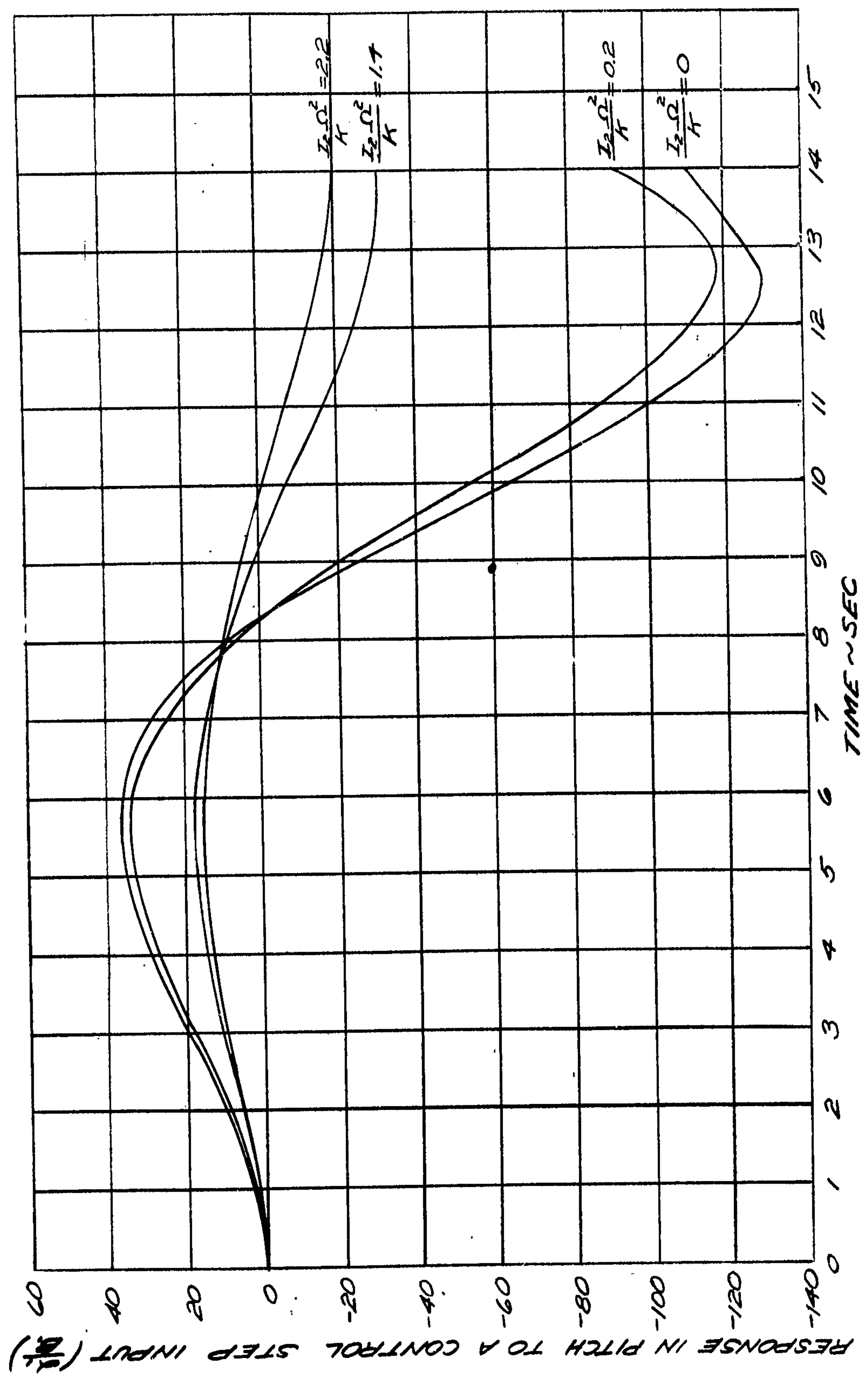

FIG. 2 Effect of $\left(\frac{I_z\Omega^2}{K}\right)$ on the Helicopter's Response in Pitch to a Control Step Input

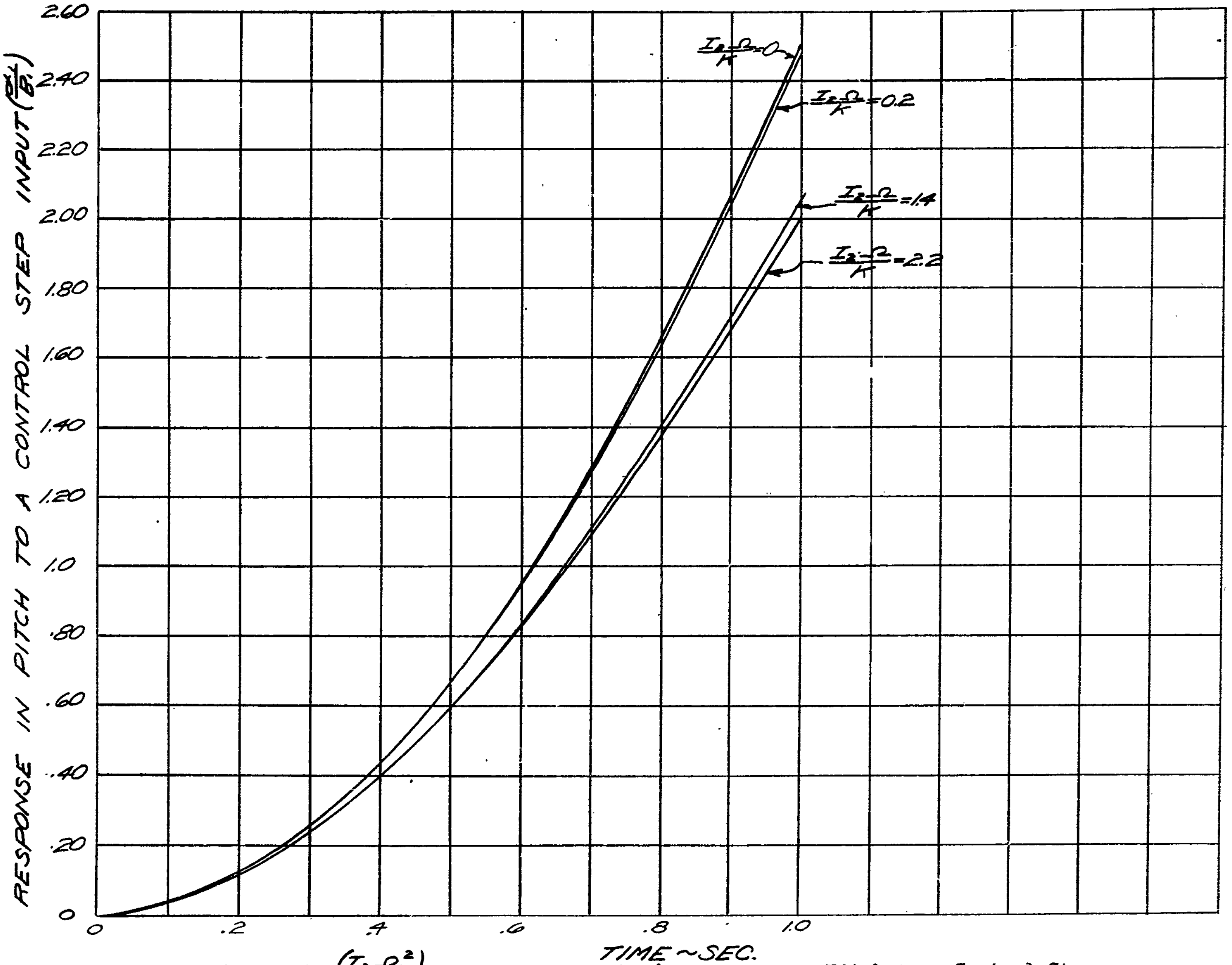

FIG. 3 Effect of $\left(\frac{I_z \Omega^2}{K}\right)$ on the Helicopter's Response in Pitch to a Control Step Input - First Second of Response

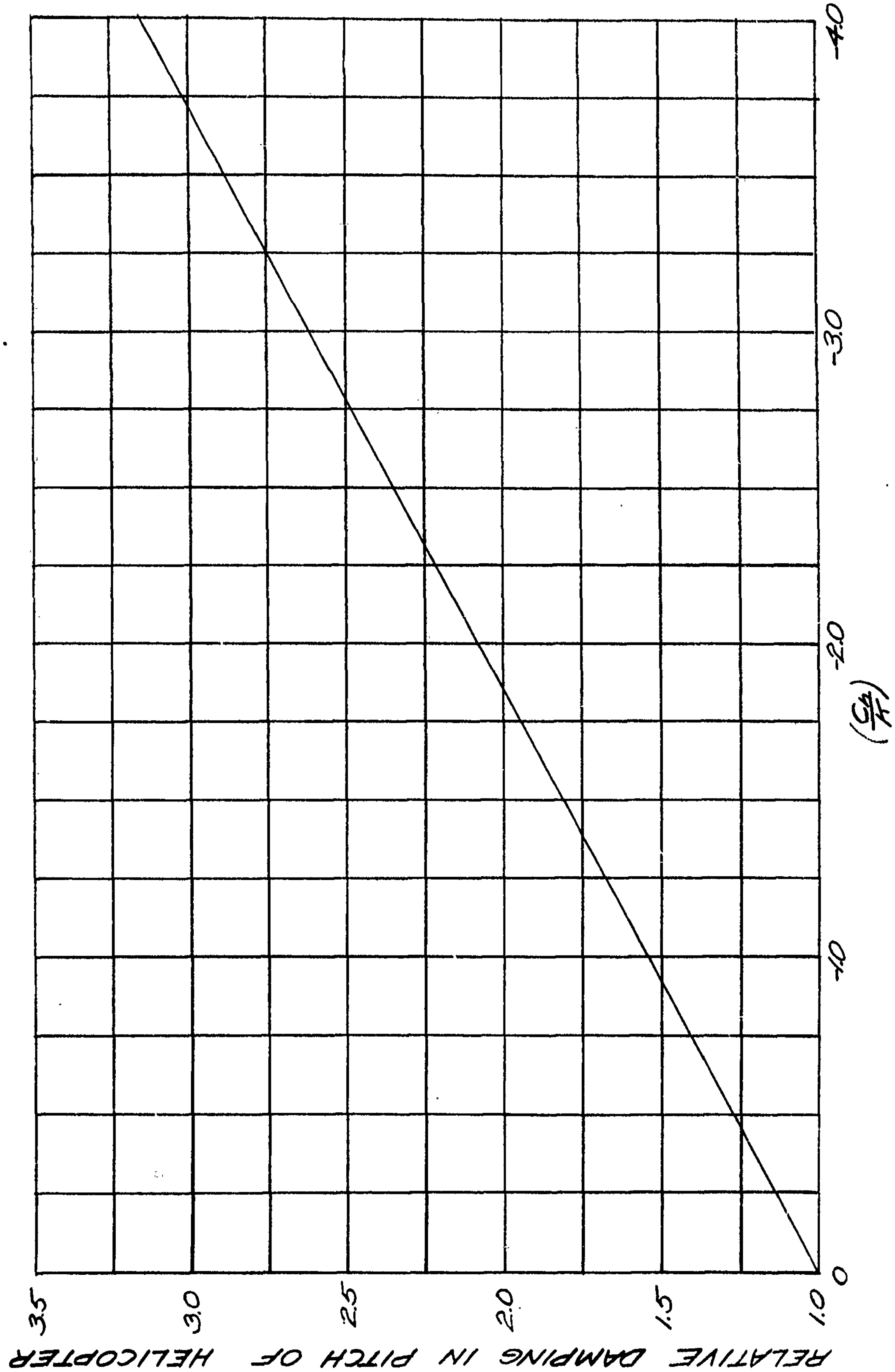

FIG. 4 Relative Damping in Pitch of Helicopter vs. $\left(\frac{C_n}{A}\right)$.

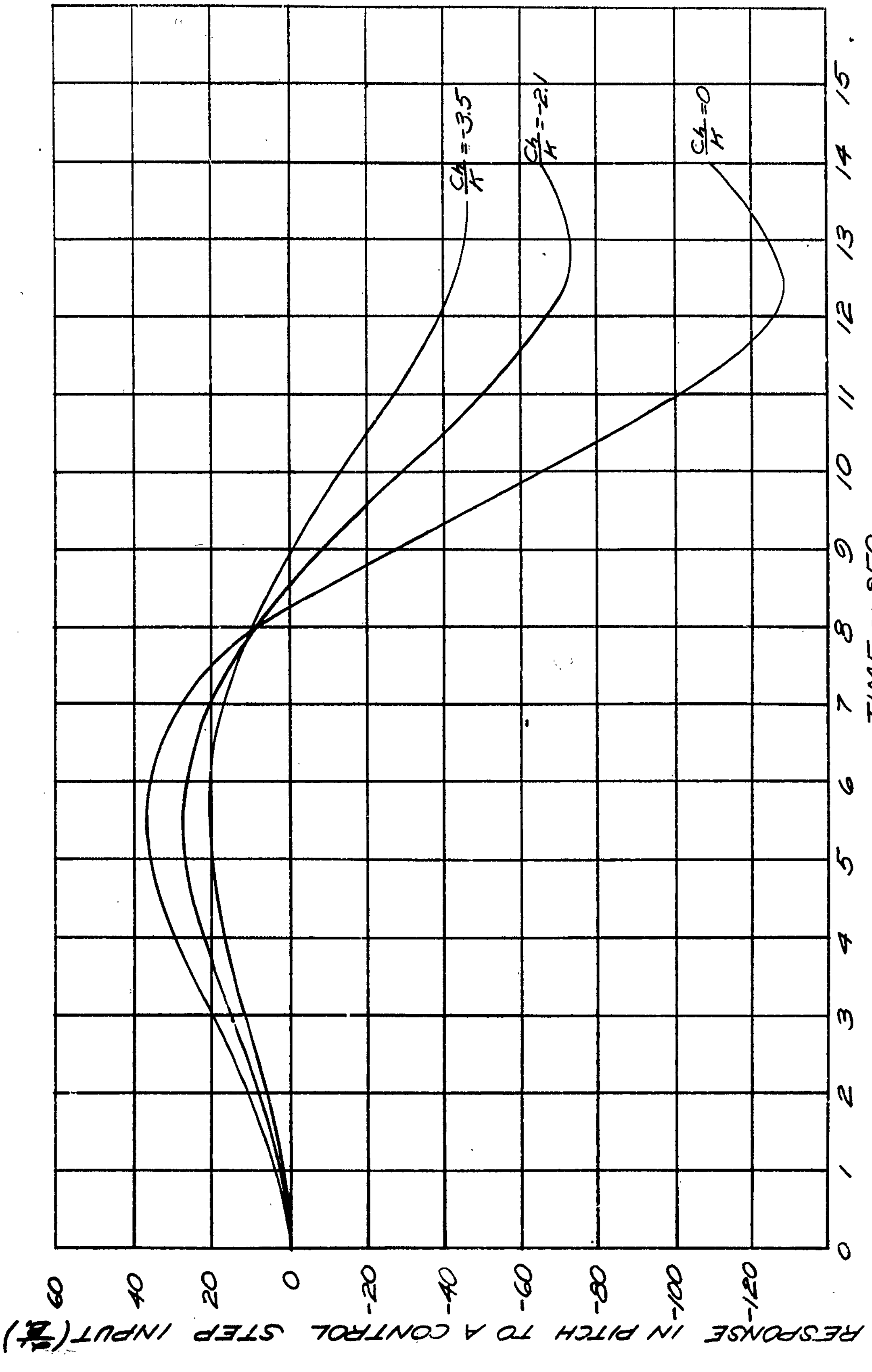

FIG. 5 Effect of $\left(\frac{C_h}{K}\right)$ on the Helicopter's Response in Pitch to a Control Step Input

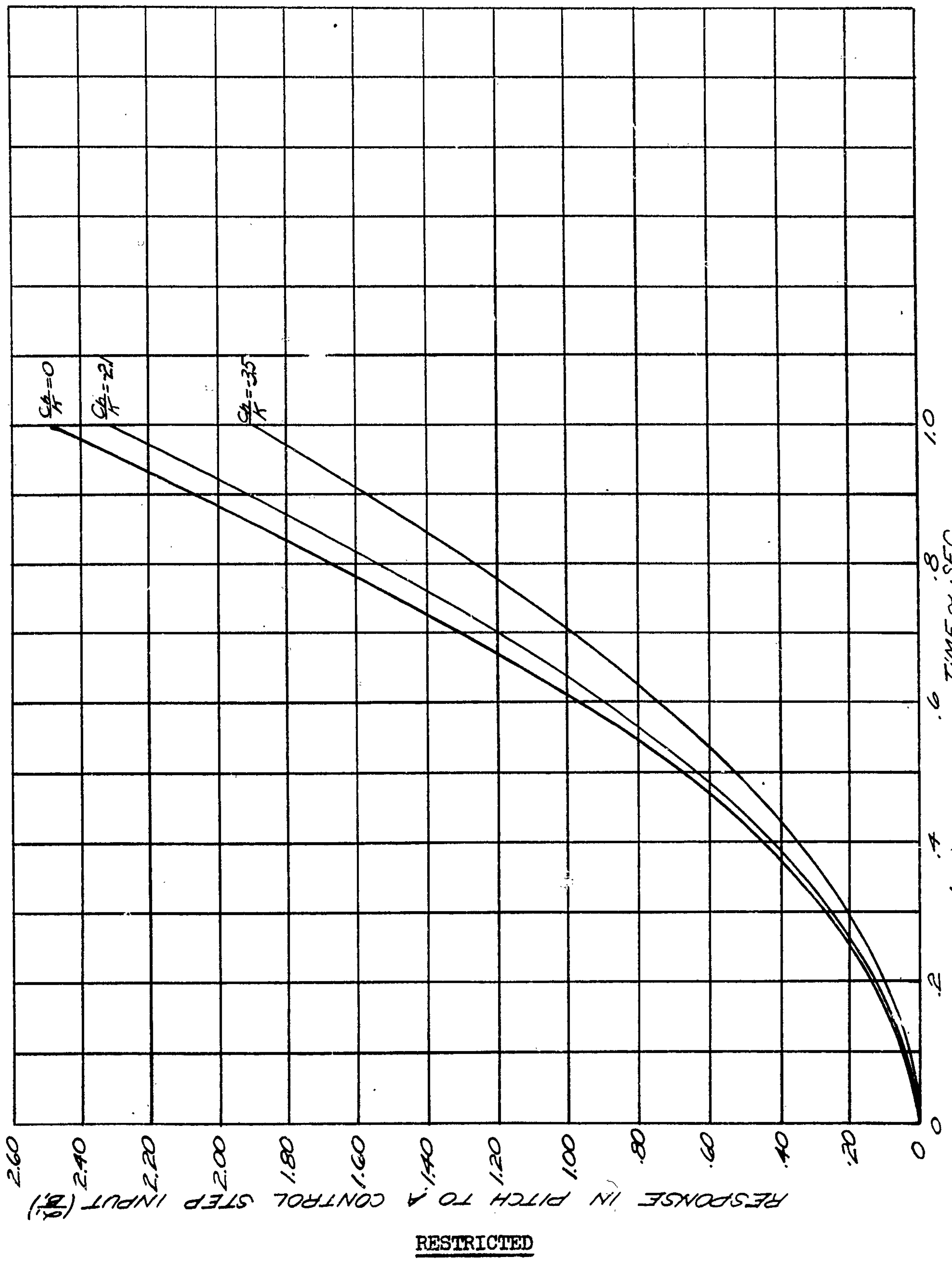

FIG. 6 Effect of $\left(\frac{C_h}{K}\right)$ on the Helicopter's Response in Pitch to a Control Step Input - First Second of Response.

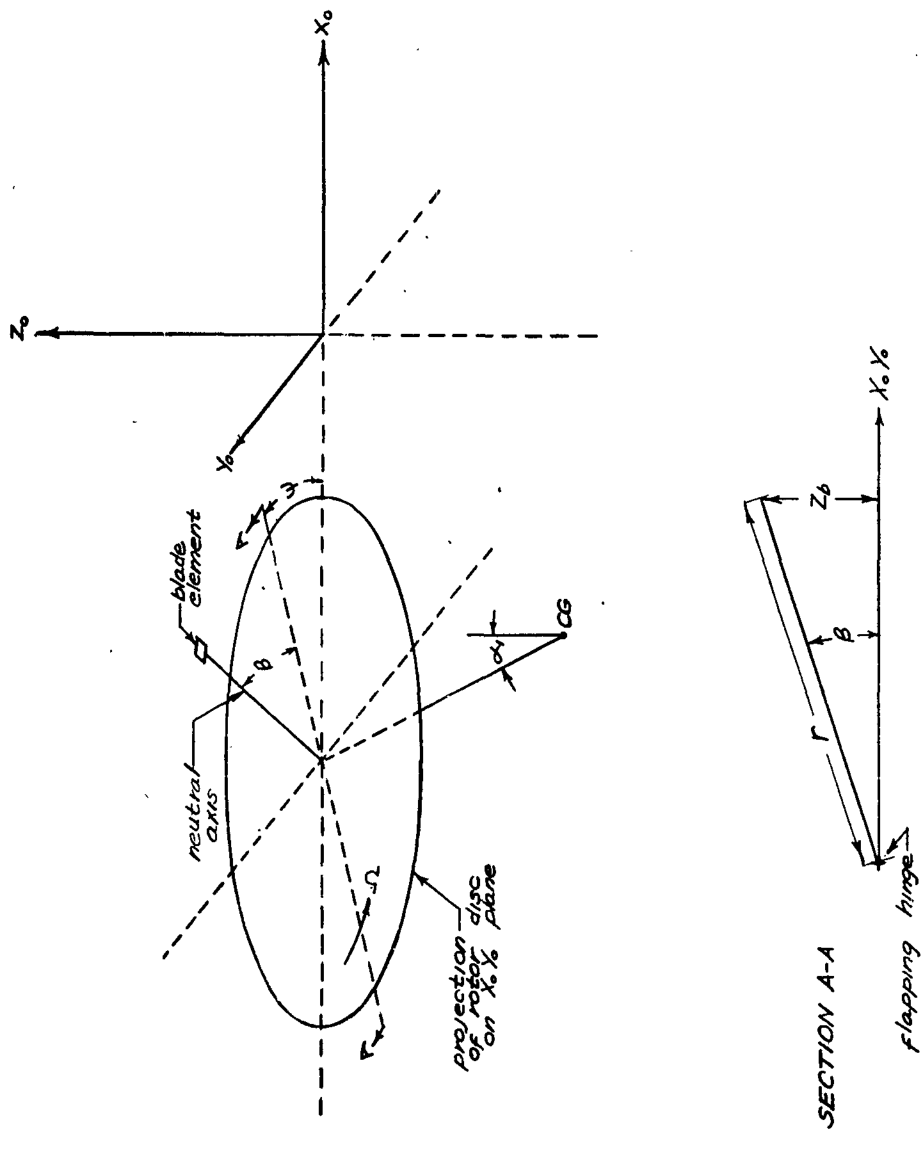

FIG. 7 Coordinate System

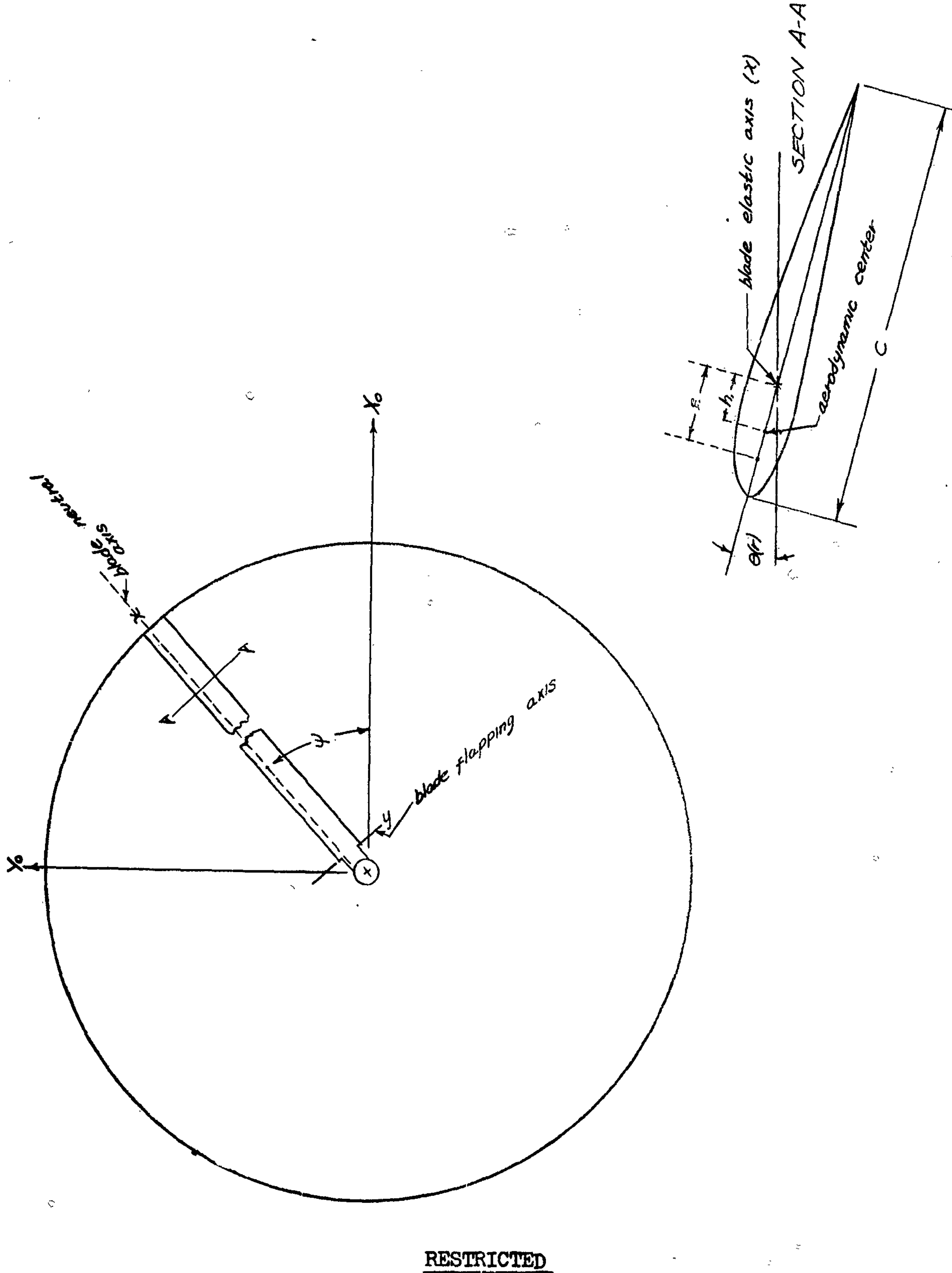

FIG. 8 - Coordinate System

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