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CONTRACT NO. N5 ori-07833 / 2
NR 064-259

025

NON-STATIONARY AERODYNAMICS OF A TWO-DIMENSIONAL BUMP IN A UNIFORM STREAM AND ITS EFFECT ON THE VIBRATION CHARACTERISTICS OF AN ELASTIC PANEL

TECHNICAL REPORT

FOR

CONTRACT N5 ori-07833

TO

OFFICE OF NAVAL RESEARCH

BY THE

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

May 1952

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AERO-ELASTIC AND STRUCTURES RESEARCH

ACKNOWLEDGMENT

The author wishes to express his gratitude to Prof. R. L. Bisplinghoff for his guidance to the research program.

Thanks are due to Messrs. G. Isakson and C. W. Brenner and Dr. T. H. H. Pian for their suggestions.

Typing was done by Miss. Eleanor Mugnai.

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1. Introduction and Summary

In dealing with the problem of the response of panels under blast loads, such as from the firing of guns or rockets, it has been observed that the severity of the damage depends upon the speed of the aircraft. From certain elementary considerations of the nature of the blast load (to be reported elsewhere), the reasons behind the phenomenon can be qualitatively explained. It is also clear, however, that the surrounding fluid might have some influence on the response through its reaction on the transient motion of the panels in question. To determine the approximate nature and the order of magnitude of such aerodynamic effects, an idealized two-dimensional configuration has been studied and the results are presented in this report.

The simplified configuration treated consists of the following: the skin structure is represented by a flat sheet of infinite length, over which a uniform stream flows at a velocity, U . The motion for each panel is considered separately, so that at one time there is only a "bump", in transient motion, on an otherwise flat sheet. The aerodynamic reaction induced by the motion of the bump is derived on the basis of small disturbances and incompressible potential flow. If there are several panels excited into motion simultaneously, the resultant aerodynamic reaction on each panel can be constructed by means of superposition.

In particular, a uniform panel with simply-supported ends is treated in detail in terms of normal coordinates. The important dimensionless parameters entering the problem are the usual reduced frequency and mass ratio (panel to air), as in all oscillating aeroelastic phenomena. The modifications on the vibration characteristics are discussed. The natural frequency in each mode tends to be reduced, as the apparent mass and the quasi-steady forces both contribute to this reduction. The reduction in the lower natural frequencies, compared with the frequencies in vacuo, is of the order of a few percent for a typical duralumin panel on an aircraft moving at an average speed, but may become large for more flexible panels and for panels on submarines under water. The higher natural frequencies are practically unaffected by the surrounding fluid.

An indication is further made for the condition for dynamic instability based upon a representation of the two lowest modes. It is found that for typical aircraft panels, instability is essentially caused by the quasi-steady forces. The aerodynamic damping makes very little contribution under ordinary circumstances.

It is hoped that, despite the over-simplification of the two-dimensional treatment, application to actual cases may be made for first order estimations. This may be done, for example, by using the concept of the "representative section" well-known in the field of aeroelasticity.

2. Method of Solution

Consider the complex potential

$$w(z) = h e^{i\alpha z} \tag{1}$$

where $z = x + iy$, and h may be complex.

It can easily be shown that, within the scope of linearized theory for small disturbances, such a function represents the disturbance potential due to a sinusoidal boundary of period $\frac{2\pi}{\alpha}$, the boundary being taken to be close to the x-axis. Regarding Eq. (1) as an elementary solution, the velocity potential for any arbitrary boundary shape can be constructed immediately by superposition, and is seen to be

$$w(z) = \int_0^\infty h(\alpha) e^{i\alpha z} d\alpha \tag{2}$$

or

$$\phi = R\ell\left\{ \int_0^\infty h(\alpha) e^{i\alpha z} d\alpha \right\} \tag{3}$$

Differentiating, we obtain

$$\phi_y = R\ell\left\{ \int_0^\infty h(\alpha)(-\alpha) e^{i\alpha z} d\alpha \right\}$$

On the boundary, therefore, the derivative is

$$\phi_y \Big|_{y=0} = R\ell\left\{ \int_0^\infty (-\alpha) h(\alpha) e^{i\alpha x} d\alpha \right\} \tag{4}$$

$$= \int_0^{\infty} -\alpha \, \frac{h(\alpha)e^{i\alpha x} + \bar{h}(\alpha)e^{-i\alpha x}}{i} \, d\alpha \tag{4a}$$

where $\bar{h}(\alpha)$ denotes the conjugate of $h(\alpha)$

If the derivative $\phi_y/_{y=0}$ is expanded by the Fourier integral, we may write

$$\phi_y/_{y=0} = \int_{-\infty}^{\infty} B(\alpha)e^{i\alpha x} \, d\alpha. \tag{5}$$

$$= 2\int_0^{\infty} \frac{B(\alpha)e^{i\alpha x} + B(-\alpha)e^{-i\alpha x}}{2} \, d\alpha \tag{5a}$$

But, with the relationship that,

$$B(-\alpha) = \bar{B}(\alpha)$$

$\phi_y/_{y=0}$ becomes

$$\phi_y/_{y=0} = \int_0^{\infty} \left[B(\alpha)e^{i\alpha x} + \bar{B}(\alpha)e^{-i\alpha x}\right] d\alpha \tag{5b}$$

Comparing Eq. (5b) with Eq. (4a), we may conclude that

$$-\alpha h(\alpha) = 2B(\alpha) \tag{6}$$

The function, $B(\alpha)$, of course is the inverse transform of $\phi_y/_{y=0}$

$$B(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_y/_{y=0} \, e^{-i\alpha x} \, dx \tag{7}$$

Hence, with the given normal velocity, $\phi_y/_{y=0}$, the disturbance potential can be evaluated.

3. Application to the Transient Motion of a Bump

In studying the problem of the transient motion of a "bump", the surfaces adjacent to the "bump" are assumed to be flat and to extend to infinity. The air stream, of velocity, U , is taken to be parallel to the flat boundary. The x-axis is chosen to coincide with the flat boundary, and the "bump" extends over in the interval, $0 \leq x \leq \pi$. Displacements of the "bump" are considered small, so that the boundary conditions are approximately satisfied on y = 0.

Thus, if y = f (x,t) represents the instantaneous shape of the bump, the boundary condition to be satisfied by the disturbance potential is:

$$\phi_y /_{y=0} = U f_x (x,t) + f_t (x,t) \qquad (8)$$

Also, in order for the disturbance to die out at large distances from the panel, the conditions

$$\phi \text{ and } \phi_y \longrightarrow 0 \text{ as } |x| \text{ or } y \text{ or both } \longrightarrow \infty \qquad (9)$$

and

$$\int_{-\infty}^{\infty} \phi_y /_{y=0} \, dx = 0 \qquad (10)$$

must be satisfied. Condition (9) is self-evident. Condition (10) represents the "closure" requirement, i.e., the instantaneous streamline must adhere to the flat boundary after passing the bump.

It is obvious that, if the instantaneous shape of the "bump" is defined by the equations

$$\left.\begin{aligned} y &= f(x,t) & 0 \le x \le \pi \\ &= 0 & \text{elsewhere} \end{aligned}\right\} \tag{11}$$

The shape may be redefined in terms of a Fourier series, so that

$$\left.\begin{aligned} y &= \sum_{n=1}^{\infty} q_n(t) \sin nx & 0 \le x \le \pi \\ &= 0 & \text{elsewhere} \end{aligned}\right\} \tag{12}$$

Here it is assumed that the bump starts from zero at $x = 0$ and $x = \pi$. If the "bump" represents a panel of homogeneous material and of constant thickness, each Fourier component corresponds to a normal mode, and $q_n(t)$ corresponds to the transient deflection in terms of the normal mode components. By introducing Eqs. (11) and (12) into Eq. (8) the boundary condition becomes

$$\left.\begin{aligned} \phi_y/_{y=0} &= U \sum_{n=1}^{\infty} n q_n(t) \cos nx + \sum_{n=1}^{\infty} q_n'(t) \sin nx \\ &= \sum_{n=1}^{\infty} A_n(t) \cos(nx - \delta_n), \quad 0 \le x \le \pi \\ \phi_y/_{y=0} &= 0 \qquad \text{elsewhere} \end{aligned}\right\} \tag{13}$$

And consequently, the problem is reduced to finding a velocity potential satisfying each term of Eq. (13), namely,

$$\left.\begin{aligned} \phi_y/_{y=0} &= A_n(t) \cos(nx - \delta_n) & 0 \le x \le \pi \\ &= 0 & \text{elsewhere} \end{aligned}\right\} \tag{13a}$$

together with conditions (9) and (10). The complete solution is then obtained by superposition.

Proceeding as indicated above, for Eq. (13a) we obtain by using

Eq. (7),

$$B_n(\alpha) = \frac{A_n}{2\pi}\int_0^{\pi} \cos(nx - \delta_n)\, e^{-i\alpha x}\, dx$$

$$= \frac{A_n}{4\pi i}\left[e^{i(n-\alpha)\pi} - 1\right]\left[\frac{2\alpha\cos\delta_n - 2in\sin\delta_n}{n^2 - \alpha^2}\right] \quad (14)$$

it follows from Eq. (6), that

$$h_n(\alpha) = -\frac{A_n}{\pi i}\left\{\cos\delta_n\left(\frac{e^{i(n-\alpha)\pi} - 1}{n^2 - \alpha^2}\right) - in\sin\delta_n\left(\frac{e^{i(n-\alpha)\pi} - 1}{\alpha(n^2 - \alpha^2)}\right)\right\} \quad (15)$$

and, finally, from Eq. (3) and with y = 0, we see that

$$\phi_n\Big|_{y=0} = -\frac{A_n}{\pi}\cos\delta_n\int_0^{\infty}\frac{-\cos n\pi . \sin\alpha(\pi - x) - \sin\alpha x}{n^2 - \alpha^2}\, d\alpha$$

$$+ \frac{A_n}{\pi}\, n\sin\delta_n\int_0^{\infty}\frac{\cos n\pi . \cos\alpha(\pi - x) - \cos\alpha x}{\alpha(n^2 - \alpha^2)}\, d\alpha \quad (16)$$

Next, the functions,

$$I_1(p,q) = \frac{1}{q}\left[Ci(pq)\sin pq - Si(pq)\cos pq\right]$$

$$I_2(p,q) = \frac{1}{q^2}\left[Ci(pq)\cos pq + Si(pq)\sin pq\right] \quad (17)$$

may be defined where $Ci(\xi)$ and $Si(\xi)$ are the cosine and sine integrals*
respectively

* These integrals are tabulated, e.g., in Ref. 2.

$$Ci(\xi) = \int_{\infty}^{\xi} \frac{\cos t \, dt}{t}$$

$$Si(\xi) = \int_{0}^{\xi} \frac{\sin t \, dt}{t} \tag{18}$$

Both integrals in Eq. (16) show singularities within the range of integration, but difficulty will only be encountered at the origin, $\alpha = 0$. Expanding Eq. (16) into partial fractions and integrating, we obtain

$$\begin{aligned} \phi_n/_{y=0} = &-\frac{A_n}{\pi} \cos \delta_n \left[-\cos n\pi \, I_1(\pi - x, n) - I_1(x, n) \right] + \\ &\frac{A_n}{\pi} n \sin \delta_n \left[\cos n\pi \, I_2(\pi - x, n) - I_2(x, n) \right] + \\ &\frac{A_n}{\pi} \frac{\sin \delta_n}{n} \lim_{\epsilon \to 0} \left\{ -\cos n\pi \quad Ci((\pi - x)\epsilon) + Ci(x\epsilon) \right\} \end{aligned} \tag{19}$$

For even values of n, we have (Ref. 2)

$$\lim_{\epsilon \to 0} \left[-Ci((\pi - x)\epsilon) + Ci(x\epsilon) \right] = \ln \left| \frac{x}{\pi - x} \right| \tag{20}$$

But for odd values of n, there is a divergence of ϕ_n even for large values of $|x|$, and thus condition (9) is not satisfied.

The source of this difficulty can be traced to the fact that the closure condition, condition (10), is not satisfied by that part of $\phi_y/_{y=0}$, specified by Eq. (13a), which is associated with $\sin \delta_n$.

Defining the function

$$\left.\begin{aligned} \phi_y\big|_{y=0} &= A_n \sin nx \sin \delta_n , \quad 0 \le x \le \pi \\ &= 0 \qquad \text{elsewhere} \end{aligned}\right\}$$

we see that

$$\left.\begin{aligned} \int_0^{\pi} A_n \sin \delta_n \sin nx \, dx &= \frac{A_n \sin \delta_n}{n} (1 - \cos n\pi) \\ &\neq 0 \text{ if } n \quad \text{is odd} \end{aligned}\right\} \tag{21}$$

Eq. (21) defines a constant displacement of the stream line in the y-direction. As shown in Appendix A, the addition of another "bump" in the form of a step eliminates this constant displacement, and the closure condition is satisfied. The divergent term in Eq. (19) then disappears. It follows, therefore, that

$$\begin{aligned} \phi_n\big|_{y=0} = &-\frac{A_n}{\pi} \cos \delta_n \left[-\cos n\pi \; I_1(\pi - x, n) - I_1(x, n) \right] + \\ &\frac{A_n}{\pi} n \sin \delta_n \left[\cos n\pi \; I_2(\pi - x, n) - I_2(x, n) + I_3 \right] \end{aligned} \tag{19a}$$

where

$$\left.\begin{aligned} I_3 &= \frac{1}{n^2} \ln \left| \frac{x}{\pi - x} \right| \qquad \text{for even } n \\ &= 0 \qquad \text{for odd } n \end{aligned}\right\} \tag{22}$$

Rewriting I_1 and I_2 according to their definitions, Eqs. (17), we obtain finally,

$$\phi_n \Big|_{y=0} = -\frac{A_{nc}}{n\pi}\Big[C_1(nx)\sin nx + S_1(nx)\cos nx\Big] +$$

$$\frac{A_{ns}}{n\pi}\Big[C_1(nx)\cos nx - S_1(nx)\sin nx\Big] +$$

$$\frac{1+\cos n\pi}{2\pi n} A_{ns} \ln\left|\frac{x}{\pi - x}\right| \tag{23}$$

where A_{nc} and A_{ns} are abbreviations for $A_n \cos \delta_n$ and $A_n \sin \delta_n$ respectively, and

$$\left.\begin{aligned} C_1(nx) &= Ci(n\pi - nx) - Ci(nx) \\ S_1(nx) &= Si(n\pi - nx) + Si(nx) \end{aligned}\right\} \tag{24}$$

Since, by definition,

$$Ci(-\xi) = Ci(\xi)$$

$$Si(-\xi) = -Si(\xi)$$

the condition that

$$(\phi_n)_{y=0} \to 0 \quad \text{as} \quad x \to \infty$$

is satisfied as required.

4. The Pressure Distribution

Having obtained the solution for ϕ_n in Eq. (23), we may now evaluate the pressure distribution along the "bump". The pressure is given by

$$\Delta p = -\rho U \phi_x \Big|_{y=0} - \rho \phi_t \Big|_{y=0} \tag{25}$$

with the pressure on the underside (y = -0), taken to be the same as that at infinity.

Differentiating Eq. (23), we obtain for the nth component,

$$\begin{aligned} \phi_{n,x}\Big|_{y=0} = &-\frac{A_{nc}}{\pi}\Big\{\left[C_1'(nx) - S_1(nx)\right]\sin nx + \\ &\qquad \left[S_1'(nx) + C_1(nx)\right]\cos nx\Big\} + \\ &\frac{A_{ns}}{\pi}\Big\{\left[C_1'(nx) - S_1(nx)\right]\cos nx - \\ &\qquad \left[S_1'(nx) + C_1(nx)\right]\sin nx\Big\} + \\ &\frac{1+\cos n\pi}{2\pi n} A_{ns}\left\{\frac{1}{|x|} - \frac{1}{|\pi - x|}\right\} \end{aligned} \tag{26}$$

where

$$\begin{aligned} C_1'(nx) &= \frac{\partial}{\partial(nx)} C_1(nx) = -\frac{\cos nx}{n}\left[\frac{\cos n\pi}{\pi - x} + \frac{1}{x}\right] \\ S_1'(nx) &= \frac{\partial}{\partial(nx)} S_1(nx) = \frac{\sin nx}{n}\left[\frac{\cos n\pi}{\pi - x} + \frac{1}{x}\right]. \end{aligned} \tag{27}$$

After simplification, in the region $0 \leq x \leq \pi$, i.e., along the "bump"

$$\phi_{n,x}\Big/_{y=0} = -\frac{A_{nc}}{\pi}\left[C_1(nx)\cos nx - S_1(nx)\sin nx\right] -$$

$$\frac{A_{ns}}{\pi}\left[C_1(nx)\sin nx + S_1(nx)\cos nx + \frac{1-\cos n\pi}{2n}\left(\frac{1}{x} - \frac{1}{\pi - x}\right)\right] \qquad (28)$$

For points beyond the "bump", a corresponding expression may readily be written, which should be used when the mutual influence of two adjacent "bumps" is to be computed. This, however, is omitted here.

Since, by definition, the following symmetry holds:

$$C_1(nx) = -C_1(n\pi - nx)$$

$$S_1(nx) = S_1(n\pi - nx)$$

it may then be verified that:

(1) for n = odd, the part of $\phi_{n,x}\Big/_{y=0}$ in Eq. (28) due to A_{nc} is symmetrical with respect to the mid-point; the part due to A_{ns} is anti-symmetrical;

(2) for n = even, the part due to A_{nc} is anti-symmetrical with respect to the mid-point; the part due to A_{ns} is symmetrical.

It will be seen that the last term in Eq. (28) involves singularities at the leading and trailing edges. This may be interpreted as being due to an effective movement of the segment, $0 \leq x \leq \pi$, as a rigid body from the flat boundary. The rest of the terms result from the deformation about this means position.

For the contribution due to ϕ_t , the same expression, Eq. (23), may be used except that the amplitudes A_{nc} and A_{ns} must be replaced by their derivatives $\dot{A}_{nc} = \frac{d}{dt}(A_{nc})$ and $\dot{A}_{ns} = \frac{d}{dt}(A_{ns})$ respectively. Thus

$$\phi_{n,t}\Big/_{y=0} = -\frac{\dot{A}_{nc}}{n\pi}\left[C_1(nx)\sin nx + S_1(nx)\cos nx\right] + \frac{\dot{A}_{ns}}{n\pi}\left[C_1(nx)\cos nx - S_1(nx)\sin nx + \frac{1+\cos n\pi}{2}\ \ln\left|\frac{x}{\pi - x}\right|\right] \tag{29}$$

We are now in a position to evaluate the total change of pressure due to the motion of the "bump":

$$\Delta p = -\rho U \sum_{n=1}^{\infty} \phi_{n,x}\Big/_{y=0} - \rho \sum_{n=1}^{\infty} \phi_{n,t}\Big/_{y=0} \tag{30}$$

with $\phi_{n,x}\Big/_{y=0}$ and $\phi_{n,t}\Big/_{y=0}$ given by Eqs. (28) and (29).

5. Generalized Forces on a Uniform Simply-Supported Panel Due to Aerodynamic Pressure

For a homogeneous panel of uniform thickness, the Fourier components of the deflection are identical with the normal modes. The coefficients, $q_n(t)$, in Eq. (12), therefore, represent the deflection in terms of normal coordinates. We may then write the equations of motion as

$$M_r \ddot{q}_r + M_r \omega_r^2 q_r = Q_r \,, \qquad r = 1,2,3,\ldots\ldots\infty \,. \tag{31}$$

where

M_r = generalized mass for the r^{th} mode,

q_r = deflection of the r^{th} mode,

ω_r = r^{th} natural frequency,

Q_r = generalized force exciting motions in the r^{th} mode.

It is well-known that Q_r may be derived from the principle of virtual work. When a pressure distribution acts on the panel, the virtual work, δW, given by a virtual displacement, δy, is

$$\delta W = \int_0^{\pi} -\Delta p \, dx \, \delta y \,. \tag{32}$$

Expressing y in terms of normal coordinates, we find that

$$y = \sum_{n=1}^{\infty} q_n \sin nx$$

$$\delta y = \sum_{n=1}^{\infty} \delta q_n \sin nx$$

Substituting into Eq. (32), we obtain

$$Q_r = \frac{\delta W}{\delta q_r} = \int_0^{\pi} -\Delta p \sin rx \, dx \tag{33}$$

The term M_r is given by

$$M_r = \mu \int_0^{\pi} \sin^2 rx\, dx = \frac{\pi}{2}\mu = \frac{1}{2} M \tag{34}$$

where

μ = mass per unit span

M = total mass of the panel

With Eq. (30) for Δp, it follows that

$$Q_r = \rho U \sum_{n=1}^{\infty} \int_0^{\pi} \phi_{n,x}\Big/_{y=0} \sin rx\, dx$$
$$+ \rho \sum_{n=1}^{\infty} \int_0^{\pi} \phi_{n,t}\Big/_{y=0} \sin rx\, dx \tag{35}$$

In using Eq. (35) we must take

$$A_{ns} = \dot{q}_n, \quad A_{nc} = U n q_n$$

as defined by Eqs. (13) and (23). For abbreviation, Eq. (35) may now be rewritten in the form:

$$\frac{Q_r}{\frac{1}{2}\rho U^2 \pi} = \sum_{n=1}^{\infty} \pi \frac{\alpha_{n,r}}{U^2} \ddot{q}_n + \sum_{n=1}^{\infty} \frac{\beta_{n,r}}{U} \dot{q}_n + \sum_{n=1}^{\infty} \frac{1}{\pi} \gamma_{n,r} q_n \tag{36}$$

and α, β, γ are identified as follows:

$$\left.\begin{aligned}
\alpha_{n,r} &= \frac{2}{n\pi^3} \int_0^{\pi} \sin rx \left[C_1(nx)\cos nx - S_1(nx)\sin nx + \frac{1+\cos n\pi}{2} \ln \frac{x}{\pi - x} \right] dx \\
\beta_{n,r} &= \frac{-4}{\pi^2} \int_0^{\pi} \sin rx \left[C_1(nx)\sin nx + S_1(nx)\cos nx + \frac{1-\cos n\pi}{4n}\left(\frac{1}{x} - \frac{1}{\pi - x}\right) \right] dx \\
\gamma_{n,r} &= -\frac{2n}{\pi} \int_0^{\pi} \sin rx \left[C_1(nx)\cos nx - S_1(nx)\sin nx \right] dx
\end{aligned}\right\} \tag{37}$$

Physically, $\alpha_{n,r}$ represents the apparent mass effect of the n^{th} mode on the r^{th} mode, $\beta_{n,r}$ represents the corresponding damping, and $\gamma_{n,r}$ represents the quasi-steady aerodynamic force.

We may now introduce the functions,

$$\left.\begin{aligned}
J_{s_1}(n,m) &= \int_0^{\pi} S_1(nx)\sin mx\,dx \\
J_{s_2}(n,m) &= \int_0^{\pi} S_1(nx)\cos mx\,dx \\
J_{c_1}(n,m) &= \int_0^{\pi} C_1(nx)\sin mx\,dx \\
J_{c_2}(n,m) &= \int_0^{\pi} C_1(nx)\cos mx\,dx
\end{aligned}\right\} \qquad (38)$$

which are evaluated in Appendix B.

Having defined the functions of Eq. (38), we may write Eq. (36) as

$$\begin{aligned}
\alpha_{n,r} = \frac{2}{n\pi^3}\Big[&\frac{1}{2} J_{c_1}(n,r-n) + \frac{1}{2} J_{c_1}(n,r+n) - \\
&\frac{1}{2} J_{s_2}(n,r-n) + \frac{1}{2} J_{s_2}(n,r+n) + \\
&\frac{1+\cos n\pi}{2}\int_0^{\pi} \sin rx \,\ln\frac{x}{\pi - x}\,dx \Big]
\end{aligned} \qquad (39)$$

$$\beta_{n,r} = -\frac{4}{\pi^2}\Big[\frac{1}{2}J_{c_2}(n, r-n) - \frac{1}{2}J_{c_2}(n, r+n) +$$

$$\frac{1}{2}J_{s_1}(n, r-n) + \frac{1}{2}J_{s_1}(n, r+n) +$$

$$\frac{1-\cos n\pi}{4n}\int_0^{\pi} \sin rx\left(\frac{1}{x} - \frac{1}{\pi - x}\right)dx\Big]$$

(39)

$$\gamma_{n,r} = -\frac{2n}{\pi}\Big[\frac{1}{2}J_{c_1}(n, r-n) + \frac{1}{2}J_{c_2}(n, r+n) -$$

$$\frac{1}{2}J_{s_2}(n, r-n) + \frac{1}{2}J_{s_2}(n, r+n)\Big]$$

Making use of formulas (B.1) to (B.12), we may write, for $n \neq r$

$$\alpha_{n,r} = \frac{2}{n\pi^3}\Big\{-\frac{r}{r^2-n^2}\left[1+\cos(r+n)\pi\right]\left[Ci(r\pi) - Ci(n\pi) + \ln\frac{n}{r}\right] +$$

$$\frac{1+\cos n\pi}{2}\left(-\frac{1+\cos r\pi}{r}\right)\left[\ln r\pi\gamma - Ci(r\pi)\right]\Big\}$$

(40)

where γ = Euler's constant = 1.781072.

Similarly, for n = r, $\alpha_{n,n}$ becomes

$$\alpha_{n,n} = \frac{2}{n\pi^3}\Big[-\pi Si(n\pi) - \frac{1}{n}(\cos n\pi - 1) - \frac{1+\cos n\pi}{n}\Big\{\ln n\pi\gamma - Ci(n\pi)\Big\}\Big] \quad (41)$$

In a like manner, $\beta_{n,r}$ may be written, for $n \neq r$

$$\beta_{n,r} = -\frac{4}{\pi^2}\Big\{\left[1-\cos(r+n)\pi\right]\frac{n\,Si(r\pi) + r\,Si(n\pi)}{r^2-n^2} + \frac{1-\cos n\pi}{4n}(1+\cos r\pi)Si(r\pi)\Big\} \quad (42)$$

and it is seen that, for n = r, $\beta_{n,n} = 0$.

Finally, the expression for $\gamma_{n,r}$ may be written, for $n \neq r$

$$\gamma_{n,r} = -\frac{2n}{\pi}\left(-\frac{r}{r^2-n^2}\right)\left(1+\cos(r+n)\pi\right)\left[Ci(r\pi) - Ci(n\pi) + \ln\frac{n}{r}\right]$$

and, for n = r, $\gamma_{n,n}$ becomes (43)

$$\gamma_{n,n} = -\frac{2n}{\pi}\left[-\pi\, Si(n\pi) + \frac{1-\cos n\pi}{n}\right]$$

We now have all the coefficients, α, β, γ 's, for evaluating the generalized force, Q_r, as defined by Eq. (36). It should be noted that:

(1) $\alpha_{n,r} \neq 0$ only when n and r are both even or both odd, i.e., the odd modes do not contribute to the apparent mass of the even modes and vice versa;

(2) $\beta_{n,r} \neq 0$ only when n and r are not both even or both add, i.e. the odd modes contribute to the damping of only the even modes and the even modes contribute only to the odd modes;

(3) $\beta_{n,n} = 0$ i.e., there is no damping of the mode due to its own motion;

(4) $\gamma_{r,n} = \gamma_{n,r}$ i.e. the quasi-steady forces are reciprocal;

(5) $\gamma_{r,n} \neq 0$ only when n and r are both even or odd, i.e. the odd modes do not contribute through their quasi-steady forces to the motion of the even modes and vice versa;

(6) The mutual effects of neighboring modes are in general larger than those of widely separated modes, owing to the factor, $r^2 - n^2$, appearing in the denominators of each coefficient;

(7) For a given r, the apparent-mass and quasi-steady forces due to other modes diminishes in the manner $\frac{\ell_{nn}}{n}$ for large $n >> r$, and as $\frac{\ell_{nr}}{r}$ for large $r >> n$.

6. Effects on the Vibration Characteristics Due to the Aerodynamic Pressure

Combining Eqs. (31) and (36), we now have a system of an infinite number of linear differential equations:

$$\sum_{n=1}^{\infty}\left\{\left(\pi\frac{\kappa}{U^2}\delta_{nr}-\frac{\pi}{U^2}\alpha_{nr}\right)q_n''-\frac{1}{U}\varrho_{nr}\dot{q}_n+\left(\pi\frac{\kappa\omega_r^2}{U^2}\delta_{nr}-\frac{\gamma_{nr}}{\pi}\right)q_n\right\}=0 \qquad r=1,2,\cdots\infty \tag{44}$$

where

δ_{nr} = Kronecker's symbol

= 1 if $n = r$

= 0 if $n \neq r$

$\kappa = \frac{M}{\rho\pi^2}$ = mass ratio

A new set of natural frequencies may now be found. Assuming that in Eq. (44),

$$q_n = \bar{q}_n e^{i\omega' t}$$

where $\bar{q}_n$ and ω' are the magnitude and frequency of the motion in the n^{th} normal coordinate, it follows that

$$\sum_{n=1}^{\infty}\left\{\left(\pi\frac{K}{U^2}\delta_{nr}-\frac{\pi\alpha_{nr}}{U^2}\right)(-\omega'^2)-i\omega'\frac{\beta_{nr}}{U}+\left(\pi\frac{K\omega_r^2}{U^2}\delta_{nr}-\frac{\gamma_{nr}}{\pi}\right)\right\}\bar{q}_n=0 \qquad r=1,2\cdots\infty \tag{45}$$

The new natural frequencies, ω', therefore, satisfy the infinite determinant

$$|A_{nr}|=0 \qquad n,r=1,2\cdots\cdots\infty \tag{46}$$

where

$$\left.\begin{aligned} A_{nr} &= -\Omega'^2(K\delta_{nr}-\alpha_{nr})-i\Omega'\beta_{nr}+K\Omega_r^2\delta_{nr}-\gamma_{nr} \\ \Omega' &= \text{modified reduced natural frequency} \\ &= \frac{\omega'\pi}{U} \\ \Omega_r &= \text{original reduced natural frequency} = \frac{\omega_r\pi}{U} \end{aligned}\right\} \tag{46a}$$

In evaluating the infinite determinant it is necessary to consider a finite number of modes. It is obvious that since the aerodynamic forces are smoothly distributed, their effect is likely to be confined only to the first few modes. For example, let us examine only the three lowest modes.

$$\Delta_3=\begin{vmatrix} -\Omega'^2(K-\alpha_{11})+K\Omega_1^2-\gamma_{11} & -i\Omega'\beta_{12} & \Omega'^2\alpha_{13}-\gamma_{13} \\ -i\Omega'\beta_{21} & -\Omega'^2(K-\alpha_{22})+K\Omega_2^2-\gamma_{22} & -i\Omega'\beta_{23} \\ \Omega'^2\alpha_{31}-\gamma_{31} & -i\Omega'\beta_{32} & -\Omega'^2(K-\alpha_{33})+K\Omega_3^2-\gamma_{33} \end{vmatrix}=0 \tag{47}$$

It is easily seen that the result will be an equation involving real numbers only, since the real and imaginary terms always occur alternately in the determinant. For $\Delta_3 = 0$ a cubic in Ω'^2 is obtained, yielding in general the three modified natural frequencies. The solution is a function of two parameters -- the mass ratio, K, and the reduced frequency, Ω_1.

From the behavior of the coupling factors between the n^{th} and the r^{th} modes, it can be shown that the contribution due to the very high modes is at most of the order, $\frac{\ln n}{n}$. Thus for the lower-mode natural frequencies, the very high modes probably may be dropped without serious error. Such of course has always been the intuitive approach for systems involving a large number of degrees of freedom. Assuming the validity of this hypothesis, we can make the further deduction that the very high natural frequencies may also be regarded as independent of the lower modes. The argument is as follows. When the r^{th} mode equation is written down for $r \gg 1$, it is seen that the very low modes have contributions of the order, $\frac{1}{n}$. Since the very low modes have been solved from a sub-determinant involving an adequate number of modes, their appearance in the equations for the very high modes may be treated as being an external forcing function, and should not influence the natural frequencies. Consequently only an adequate sub-determinant involving modes of the same order as the r^{th} mode is needed to evaluate the r^{th} natural frequency. Besides γ_{rr}, only β_{nr}'s exist for large r, and are of the order, $\left|\frac{1}{r-n}\right|$.

In fact, the diagonal terms, A_{nn}, are the dominant terms under ordinary circumstances, when the aerodynamic reactions are relatively small.

Without the aerodynamic reactions, all terms except those on the diagonal vanish. Now for large values of r,

$$A_{rr} \approx -\kappa \Omega'^2 + \kappa \Omega_r^2 - \gamma_{rr}$$

$$A_{nr} \approx i \beta_{nr}$$

Since Ω_r^2 is of the order of r^4, and γ_{rr} is of the order of r, while β_{nr} is at most of the order unity, the constant parts in the diagonal terms become very large in comparison with those in terms off the diagonal. It is apparent that if Ω' is evaluated from the diagonal terms only, an adequate approximation for the higher modes may be obtained.

The correction to this approximation, if desired, can then be worked out by a procedure developed by Lord Rayleigh:* Assuming that the system is excited in the modified rth mode (which is almost the original rth mode, since the coupling terms are assumed to be small), and neglecting second order terms, we have:

from the equation for the nth mode,

$$A_{nn} \bar{q}_n + A_{rn} \bar{q}_r = 0$$

from the equation for the rth mode,

$$\sum_{n=1}^{\infty}{}' A_{nr} \bar{q}_n + A_{rr} \bar{q}_r = 0 \tag{48}$$

where $\sum'$ indicates that n = r is excluded. It follows immediately, then, that

$$A_{rr} \approx \sum_{n=1}^{\infty} \frac{A_{nr} A_{rn}}{A_{nn}} \tag{49}$$

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* Ref. 3, pp. 113 - 115, also pp. 136 - 137

The right-hand side of Eq. (49) is already a second-order quantity. For a first approximation, therefore,

$$A_{rr} = -(\kappa - \alpha_{rr})\Omega'^2 + \kappa \Omega_r^2 - \gamma_{rr} \approx 0$$

whence we see that

$$\Omega'^2_{r_1} \approx \frac{\kappa}{\kappa - \alpha_{rr}} \Omega_r^2 \left[1 - \frac{\gamma_{rr}}{\kappa \Omega_r^2}\right] \tag{50}$$

The value of Ω'_{r_1} can then replace the unknown Ω' in the right-hand side of Eq. (49), and a second approximation is obtained.

$$\Omega'^2_{r_2} \approx \frac{\kappa}{\kappa - \alpha_{rr}} \Omega_r^2 \left[1 - \frac{\gamma_{rr}}{\kappa \Omega_r^2}\right] - \sum_{n=1}^{\infty}{}' \frac{(\Omega'^2_{r_1}\alpha_{nr} - i\Omega'_{r_1}\beta_{nr} - \gamma_{nr})(\Omega'^2_{r_1}\alpha_{rn} - i\Omega'_{r_1}\beta_{rn} - \gamma_{rn})}{(\kappa - \alpha_{nn})(\kappa - \alpha_{rr})(\Omega'^2_{n_1} - \Omega'^2_{r_1})} \tag{50a}$$

Thus the correction terms are mainly those due to the neighboring modes $n \approx r$

For $r \gg 1$, the important result here is that the modified natural frequency is given by

$$\frac{\omega'_r}{\omega_r} \approx \left(1 - \frac{\gamma_{rr}}{2\kappa \Omega_r^2}\right) \tag{50b}$$

Since γ_{rr} is always positive, the natural frequency is reduced. The percentage reduction decreases as r^{-3}, and may be neglected for the high modes. The reduction is also inversely proportional to the mass ratio of the panel to air. As altitude increases, κ increases, and the reduction in the natural frequency becomes smaller and smaller.

It may be noted that Eq. (49) and the succeeding approximation procedure may also be used for the lower modes, provided that the diagonal terms remain dominant. Such would be the case when K and/or Ω_1^2, is large in comparison with the α_{nr}, β_{nr} and γ_{nr} coefficients.

7. Further Investigations and Some Numerical Results Based on the First Two Modes

It is desirable to be able to indicate the order of magnitude of the aerodynamic effect on the vibration characteristics of the lower modes, following the general remarks on the high modes in the previous section. For this purpose let us consider only the first two modes. The coefficients α_{11}, etc. are found to be

$$\alpha_{11} = -.246 \,, \quad \alpha_{12} = \alpha_{21} = 0 \,, \quad \alpha_{22} = -.222 \,;$$

$$\beta_{11} = 0 \,, \quad \beta_{12} = -1.953 \,, \beta_{21} = 1.382 \,, \quad \beta_{22} = 0 \,;$$

$$\gamma_{11} = 2.43 \,, \quad \gamma_{12} = \gamma_{21} = 0 \,; \quad \gamma_{22} = 5.68 \,.$$

Thus the frequency equation becomes

$$\left[-\Omega'^2(K+.246)+K\Omega_1^2-2.43\right]\left[-\Omega'^2(K+.222)+K\Omega_2^2-5.68\right] + \Omega'^2(-1.953)(1.382)=0 \qquad (51)$$

Noting that for simply-supported ends, $\Omega_2 = 4\Omega_1$, we obtain after algebraic manipulation,

$$\Omega'^4(K+.246)(K+.222)-\Omega'^2\left[(17K+4.16)K\Omega_1^2-8.11K+0.76\right]$$
$$+16(K\Omega_1^2-.355)(K\Omega_1^2-2.43)=0 \qquad (51a)$$

Neglecting aerodynamic forces corresponds to the case of $K\to\infty$, for which the two roots of Ω'^2 reduce to Ω_1^2 and $16\Omega_1^2$, as required. For large values of K, the roots can be expressed in descending powers of K, in the manner

$$\left.\begin{aligned}\left(\frac{\omega_1'}{\omega_1}\right)^2 &\approx 1-\frac{1}{K}\left(.253+\frac{2.42}{\Omega_1^2}\right)+0\left(\frac{1}{K^2}\right)\\ \left(\frac{\omega_2'}{\omega_2}\right)^2 &\approx 1-\frac{1}{K}\left(.190+\frac{.355}{\Omega_1^2}\right)+0\left(\frac{1}{K^2}\right)\end{aligned}\right\} \qquad (52)$$

ω_1' and ω_2' being the modified first and second natural frequencies resulting from the presence of the surrounding fluid. It should be pointed out that, even for the first two modes, the natural frequencies still tend to be reduced. For the higher modes the same conclusion was reached above.

When Ω_1^2 is large, Eq. (51a) may be simplified by dropping terms of the order of $\frac{1}{\Omega_1^2}$. Thus, for $\Omega_1^2 >> 1$

$$\left(\frac{\omega'}{\omega_1}\right)^4(K+.246)(K+.222)-\left(\frac{\omega'}{\omega_1}\right)^2K(4.16+17K)+16K^2=0 \qquad (53)$$

For a first approximation, the frequency ratio, $\frac{\omega'}{\omega_1}$, is, then, a function only of the mass ratio, K. The physical interpretation of this is that the modification is now mainly due to the apparent mass of the fluid associated with the panel vibration. In contrast, the previous correction for the case, $K \gg 1$, may be regarded to originate largely from the quasi-steady aerodynamic forces.

For numerical examples, let us consider some typical values.

(1) Duralumin panel, 12" between supports, .064" thick, sea level atmosphere, U = 500 ft/sec.

$$\mu \approx .026 \text{ slugs/ft}$$

$$\omega_1 \approx 870 \text{ rad/sec}$$

$$K \approx 10.9$$

$$\Omega_1 \approx \frac{870 \times 12}{500 \times 12} = 1.74$$

From Eqs. (52), we obtain

$$\left(\frac{\omega_1'}{\omega_1}\right)^2 \approx 0.90 \quad , \quad \left(\frac{\omega_2'}{\omega_2}\right)^2 \approx 0.97$$

i.e.,

$$\frac{\omega_1'}{\omega_1} \approx .95 \quad , \quad \frac{\omega_2'}{\omega_2} \approx .98$$

(2) Steel panel, 24" between supports, ½" thick, in water, U = 30 ft/sec

$$\mu \approx .32 \text{ slugs/ft}$$

$$\omega_1 \approx 2,600 \text{ rad/sec}$$

$$K \approx .08$$

$$\Omega_1 \approx \frac{2600 \times 2}{30} = 180$$

Eq. (53) becomes

$$1.06\left(\frac{\omega'}{\omega}\right)^4 - 4.43\left(\frac{\omega'}{\omega}\right)^2 + 1.02 = 0$$

from which it follows that

$$\frac{\omega_1'}{\omega_1} \approx .50 \quad , \qquad \frac{\omega_2'}{\omega_2} \approx .50$$

A remark may be inserted here regarding the applicability of formulas derived in this report to panels in water and moving at relatively slow speeds. The theory, of course, is still valid, provided only that the deflections be sufficiently small for the linearization of the boundary conditions. The typical values in the above example, on the other hand, seem to indicate that κ will be small and Ω_1^2 large. If κ is small, the natural modes in vacuo are poor approximations to the true mode shapes. Strong coupling effects, therefore must be expected. Probably more than two modes are necessary even for a rough estimate of the modified first natural frequency.

Some discussions on the possibility of dynamic instability of the panel may also be made on the basis of the two-mode representation. Let us consider again Eq. (51a). It is apparent that a divergent motion sets in, if the value of Ω'^2 satisfying Eq. (51a) becomes negative or complex. A locus in the $\kappa - \Omega_1^2$, plane can be drawn to separate the stable and unstable regions in terms of these two parameters. Regarding Eq. (51a) as of the typical form

$$A x^2 - B x + C = 0$$

We see that the conditions for positive, real roots of x are

(1) A, B, C > 0

(2) $B^2 > 4AC$

Since $K > 0$, the coefficient of Ω'^4 is always positive. Hence the conditions for stability reduce to

(a) $$K\Omega_1^2 \geq \frac{8.11K - .76}{17K + 4.16}$$

$$\geq 0.477 + O\left(\frac{1}{K}\right) \text{ for } K >> 1 \qquad (54)$$

(b) $$(K\Omega_1^2 - .355)(K\Omega_1^2 - 2.43) \geq 0$$

i.e. $.355 \geq K\Omega_1^2$ or $K\Omega_1^2 \geq 2.43$ (55)

(c) $$\left[(17K + 4.16)K\Omega_1^2 - 8.11K + .76\right]^2 \geq$$

$$64(K + .246)(K + .222)(K\Omega_1^2 - 2.43)(K\Omega_1^2 - .355) \qquad (56)$$

Instead of tracing the complete boundary defined by Eq. (56), a simplified criterion may be derived for the case $K >> 1$. Eq. (56) is then approximated by

$$\left[\left(1 + \frac{.246}{K}\right)K\Omega_1^2 - .477 + \frac{.045}{K}\right]^2 \geq .222\left(1 + \frac{.468}{K}\right)(K\Omega_1^2 - 2.43)(K\Omega_1^2 - .355) \qquad (56a)$$

Within slide-rule accuracy, the result is

$$K\Omega_1^2 \geq .215 + \frac{.328}{\sqrt{K}} + O\left(\frac{1}{K}\right) \qquad (57)$$

Combining the conditions (54), (55) and (57), we conclude that for $K >> 1$, stability prevails if

$$K\Omega_1^2 \geq 2.43 \tag{58}$$

Physically, this means that the quasi-steady forces tend to reduce the effective spring constant in each mode. The strongest influence is on the first mode. When the effective spring constant of the first mode is reduced to a negative value, divergent motion will naturally result. The same influence on the higher modes is much smaller. For $K \gg 1$, furthermore, the damping and apparent mass terms are of no consequence in the determination of the stability range.

For a numerical example, let us take a duralumin panel of .064" thick and 48" between supports, at sea-level atmosphere. One finds

$$\mu \approx .026 \text{ slugs/ft}$$

$$\omega_1 \approx 55 \text{ rad/sec}$$

$$K \approx 2.7$$

The critical value of Ω_1^2 is, by Eq. (58)

$$(\Omega_1)^2_{cr.} = \frac{2.43}{2.7} = 0.9$$

Hence, the "critical speed" is

$$V_{cr.} = \frac{\omega_1 L}{(\Omega_1)_{cr.}} \approx 230 \text{ ft/sec.}$$

It must, however, be stressed that the results in this section can be valid only when a two-mode representation is adequate to describe panel vibration. The discussion on stability is further restricted by the assumption of small disturbances. Whether the stability involving finite deflections has roughly the same criterion or not, is a question beyond the scope of linearized theory. It may further be noted that the dynamic instability described above is different from the conventional "flutter" phenomenon. In "flutter", we may recall that the resultant damping of the system vanishes at the critical speed, while here the resultant spring constant is the criterion, at least for $K \gg 1$ (relatively heavy panel).

8. Summary of Results.

(1) Based upon the linearized theory for small disturbances, the velocity potential of arbitrary transient motions of a two-dimensional bump, on an otherwise flat surface, in a uniform incompressible stream has been determined in terms of the Si - and Ci - functions.

(2) Application of the theory is made to the problem of a simply-supported uniform panel. The effect on the vibration characteristics depends on the usual parameters, $\Omega_1 = \frac{\omega_1 L}{v}$ (the reduced frequency of the first natural mode) and $K = \frac{\mu}{\rho L}$ (the mass ratio of panel to air). Coefficients representing the apparent mass, damping and quasi-steady forces are obtained, in a form for use with normal coordinates, which are strictly true for a simply-supported panel in vacuo.

(3) For the higher modes, only the quasi-steady aerodynamic forces have some contributions. The result is a lower natural frequency. The percentage reduction, however, decreases very rapidly for higher and higher

modes. It is, furthermore, inversely proportional to the mass ratio, K, and the square of the reduced frequency, Ω_1.

(4) An iterative procedure, following Rayleigh, is indicated for the evaluation of modified natural frequencies when K and/or $\Omega_1^2 >> 1$. Panels on aircraft usually belong to the category of large K, while panels on submarines in water may correspond to the latter case of $\Omega_1^2 >> 1$.

(5) Panel behavior based on a representation of the first two modes is studied in more detail. Approximate formulas are given for cases when $K >> 1$ and when $\Omega_1^2 >> 1$. It is again found that the natural frequencies tend to diminish in the presence of the surrounding fluid.

(6) A criterion for dynamic stability is also derived based on the two-mode representation, namely, for stability

$$K \Omega_1^2 \geq 2.43$$

To illustrate, a duralumin panel of 4' span and .064" thick is estimated to have a critical speed of about 230 ft/sec at sea-level.

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(1) Goldberg, J. E., "The Measurement of Gun Muzzle Blast Pressures on a High-Speed Fighter Airplane", Air Force Technical Data Digest, Aug. 1951, 16-20.

(2) Jahnke, E, and Emde, F., "Tables of Functions with Formulae and Curves", 4th edition, Dover Publications, 1945.

(3) Lord Rayleigh, "Theory of Sound", Vol. 1, Dover Publication, 1945.

APPENDIX A

Modification on ϕ_n for Odd n, to Satisfy Conditions at Infinity

Let us consider the bump

$$\left.\phi_y\right|_{y=0} = \begin{cases} A_n \sin nx \quad \sin \delta_n & \text{for } 0 \leq x \leq \pi \\ = 0 & \text{elsewhere} \end{cases} \tag{A1}$$

where n is an odd integer. Eq. (21) shows that the closure condition (9) is violated. The streamline does not return to the flat boundary after passing the bump. It is obvious, however, that the streamline could be made to return to the flat boundary by the introduction of discontinuities. Since the condition (A1) results from the transient motion of a sine wave boundary, the general configuration is symmetrical about the mid-point ($x = \frac{\pi}{2}$). The discontinuities must bear the same kind of symmetry; and, furthermore, can only occur at the end points $x = 0$ and $x = \pi$.

The significance of these seemingly artificial discontinuities probably could be better understood from a different point of view. Regarding (A1) as the description of the slope along a bump in steady flow, one sees immediately that the shape of the hypothetical bump must be a cosine wave. If an up-and-down displacement of the cosine wave is made, two discontinuities are brought in at the end point. These discontinuities being limited to a very small segment, will not alter the condition (A1), but will serve to change the streamline directions quite abruptly in rising from and returning back to the flat boundary.

To determine the effect of these discontinuities on the flow by means of the linearized theory, a limiting process is necessary. Let Eq. (A1) first be modified into

$$\phi_y\big|_{y=0} = A_n \sin \delta_n \sin nx \;, \quad 0 \le x \le \pi$$
$$= \frac{A_n'}{\delta} \begin{cases} -\delta \le x < 0 \\ \pi < x \le \pi + \delta \end{cases}$$
$$= 0 \qquad \text{elsewhere} \tag{A2}$$

where δ is a small quantity. The magnitude, A_n', is chosen to satisfy the closure condition (9):

$$\int_{-\delta}^{0} \frac{A_n'}{\delta} dx + \int_0^{\pi} A_n \sin \delta_n \sin nx \, dx + \int_{\pi}^{\pi+\delta} \frac{A_n'}{\delta} dx = 0$$

hence

$$A_n' = -\frac{A_n \sin \delta_n}{n} \tag{A3}$$

remembering that n is odd. Using Eq. (7), the increment to $B(\alpha)$ becomes

$$\Delta B(\alpha) = \frac{1}{2\pi}\left(\int_{-\delta}^{0} + \int_{\pi}^{\pi+\delta}\right)\left(\frac{A_n'}{\delta} e^{-i\alpha x} dx\right)$$

$$= -\frac{A_n'}{2\pi i \delta \alpha}\left(1 - e^{-i\alpha\pi} + e^{-i\alpha(\pi+\delta)} - e^{i\alpha\delta}\right) \tag{A4}$$

Then by Eq. (6), we have

$$\Delta h(\alpha) = -\frac{2\Delta B(\alpha)}{\alpha} = \frac{A_n'}{i\pi\delta\alpha}\left[1 - e^{i\alpha\delta} - e^{-i\alpha\pi}(1 - e^{-i\alpha\delta})\right] \tag{A5}$$

Taking the limit, $\delta \to 0$

$$\Delta h(\alpha) = \frac{A_n'}{i\pi\alpha^2}\left[-i\alpha - e^{-i\alpha\pi}(i\alpha)\right]$$

$$= -\frac{A_n'}{\pi\alpha}\left(1 + e^{-i\alpha\pi}\right) \tag{A6}$$

Substituting into Eq. (3), we find that

$$\Delta\phi_n = Rl\left\{\int_0^\infty \Delta h(\alpha) e^{i\alpha x} d\alpha\right\} = -\frac{A_n'}{\pi}\int_0^\infty Rl\left(\frac{e^{i\alpha x} + e^{-i\alpha(\pi - x)}}{\alpha}\right) d\alpha$$

$$= -\frac{A_n'}{\pi}\int_0^\infty \frac{\cos\alpha x + \cos\alpha(\pi - x)}{\alpha} d\alpha = \frac{A_n \sin\delta_n}{n\pi}\int_0^\infty \frac{\cos\alpha x + \cos\alpha(\pi - x)}{\alpha} d\alpha$$

$$= -\frac{A_n \sin\delta_n}{n\pi} \lim_{\epsilon \to 0}\left[C_i(\epsilon x) + C_i(\epsilon)(\pi - x)\right] \tag{A7}$$

when (A7) is added to Eq. (19), the divergent terms for odd n cancel. The expression given by Eq. (19a) is thus established.

APPENDIX B

Evaluation of Integrals Appearing in the Generalised Force

The integrals introduced in Eq. (38) will be evaluated in this appendix:

(1) $J_{S_1}(n,m) = \int_0^{\pi} S_1(nx) \sin mx \, dx$

(2) $J_{S_2}(n,m) = \int_0^{\pi} S_1(nx) \cos mx \, dx$

(3) $J_{C_1}(n,m) = \int_0^{\pi} C_1(nx) \sin mx \, dx$

(4) $J_{C_2}(n,m) = \int_0^{\pi} C_1(nx) \cos mx \, dx$

(5) $\int_0^{\pi} \sin rx \, \ln \frac{x}{\pi - x} \, dx$

(6) $\int_0^{\pi} \sin rx \left(\frac{1}{x} - \frac{1}{\pi - x} \right) dx$

(1) $J_{S_1}(n,m)$

By the definition of $S_1(nx)$, we have

$$J_{S_1}(n,m) = \int_0^{\pi} \left\{ \int_0^{n(\pi - x)} + \int_0^{nx} \right\} \frac{\sin t \, dt}{t} \sin mx \, dx$$

Changing the order of integration,

$$J_{S_1}(n,m)=\int_0^{n\pi}\left\{\int_0^{\pi-\frac{t}{n}}+\int_{\frac{t}{n}}^{\pi}\right\}\sin mx\,dx\ \frac{\sin t\,dt}{t}$$

$$=\frac{1}{m}\int_0^{n\pi}\left\{1-\cos\left(m\pi-\frac{mt}{n}\right)+\cos\frac{mt}{n}-\cos m\pi\right\}\frac{\sin t\,dt}{t}$$

$$=\frac{1-\cos m\pi}{m}\int_0^{n\pi}\left\{\frac{\sin t}{t}+\frac{\cos\frac{mt}{n}\sin t}{t}\right\}dt$$

$$=\frac{1-\cos m\pi}{m}\left\{Si(n\pi)+\frac{1}{2}Si((m+n)\pi)\pm\frac{1}{2}Si(\pm(n-m)\pi)\right\} \tag{B1}$$

where the upper signs in the last term are used when $n > m$, and the lower signs are used when $n < m$. In particular,

$$J_{S_1}(n,0)=0$$

(2) $J_{S_2}(n,m)$

In a similar manner,

$$J_{S_2}(n,m)=\int_0^{\pi}\left\{\int_0^{n(\pi-x)}+\int_0^{n\pi}\right\}\frac{\sin t\,dt}{t}\cos mx\,dx$$

$$=-\frac{1+\cos m\pi}{m}\int_0^{n\pi}\frac{\sin\frac{m}{n}t\,\sin t}{t}\,dt$$

after changing the order of intergration. The integral being a divergent one, a limiting procedure must be adopted. Thus for $n \neq m$, $m \neq 0$, we have

$$J_{S_2}(n,m) = -\frac{1+\cos m\pi}{2m} \lim_{\epsilon\to 0} \int_\epsilon^{n\pi} \frac{\cos(1-\frac{m}{n})t - \cos(1+\frac{m}{n}t)}{t} dt$$

$$= -\frac{1+\cos m\pi}{2m} \lim_{\epsilon\to 0} \left[\int_\epsilon^\infty + \int_\infty^{n\pi} \right]$$

$$= -\frac{1+\cos m\pi}{2m} \Big[Ci(|n-m|\pi) - Ci((n+m)\pi) +$$

$$\lim_{\epsilon\to 0} \left\{ -Ci(|1-\tfrac{n}{m}|\epsilon) + Ci((1+\tfrac{n}{m})\epsilon) \right\} \Big]$$

$$= -\frac{1+\cos m\pi}{2m} \left[Ci(|n-m|\pi) - Ci((n+m)\pi) + \ln \left|\frac{m+n}{m-n}\right| \right]$$

(B2)

The logarithmic term is the proper limit of the sum in the curly bracket, since the asymptotic behavior of Ci is known (Ref. 2).

$$Ci(x) \sim + \ln \gamma x, \quad x << 1 \qquad \text{(B3)}$$

γ being Euler's Constant, equal to 1.78107

Similarly for $n = m$, $n \neq 0$

$$J_{S_2}(n,n) = -\frac{1+\cos n\pi}{n} \int_0^{n\pi} \frac{\sin^2 t}{t} dt$$

$$= -\frac{1+\cos n\pi}{2n} \lim_{\epsilon\to 0} \int_\epsilon^{n\pi} \frac{1-\cos 2t}{t} dt$$

Using the asymptotic behavior given by Eq. (B3), we finally obtain

$$J_{s_2}(n,n) = -\frac{1+\cos n\pi}{2n}\left[-Ci(2n\pi) + \ln 2\gamma n\pi\right] \tag{B4}$$

Another special case results when m = 0. The result is readily obtained by the same procedure,

$$J_{s_2}(n,0) = 2\pi\, Si(n\pi) + \frac{2}{n}(\cos n\pi - 1) \tag{B5}$$

(3) $J_{c_1}(n,m)$

$$J_{c_1}(n,m) = \int_0^{\pi}\left\{\int_{\infty}^{n(\pi-x)} - \int_{\infty}^{nx}\right\}\frac{\cos t}{t}\,dt\,\sin mx\,dx$$

$$= \int_0^{\pi}\left\{\int_0^{n(\pi-x)} - \int_0^{nx}\right\}\frac{\cos t}{t}\,dt\,\sin mx\,dx$$

At this point we must interpret the divergent integrals, $\int_0^{n(\pi-x)}$ and $\int_0^{nx}$, by their proper limiting values. The limiting value of the combination in the curly bracket must exist because it is simply $\int_{nx}^{n(\pi-x)}$, though it is artificially split into two terms. The integration may next be carried out easily after a change of the order of integration as was done for J_{s_2}.

The result is, for $n \neq m$, $m \neq 0$,

$$J_{c_1}(n,m) = -\frac{1+\cos m\pi}{2m}\Big\{Ci((n+m)\pi) + Ci(|n-m|\pi)$$

$$-2\,Ci(n\pi) + \ln\left|\frac{n^2}{n^2-m^2}\right|\Big\} \tag{B6}$$

for $n = m, \quad m \neq 0$

$$J_{C_1}(n,n) = -\frac{1+\cos n\pi}{2n}\left\{Ci(2n\pi) - 2\,Ci(n\pi) + \ln\frac{\gamma n\pi}{2}\right\} \tag{B7}$$

and, for $m = 0$

$$J_{C_1}(n,0) = 0$$

(4) $J_{C_2}(n,m)$

$J_{C_2}(n,m)$ involve terms which appear in J_{S_1}. The result is,

for $m \neq 0$

$$J_{C_2}(n,m) = \frac{1-\cos m\pi}{2m}\left\{Si\big((n+m)\pi\big) \mp Si\big(\pm(n-m)\pi\big)\right\} \tag{B8}$$

where in the last term the upper signs are used when $n > m$ and lower signs are used when $n < m$. For $m = 0$, $J_{C_2}(n,0) = 0$.

(5) $\int_0^\pi \sin rx \ln \frac{x}{\pi - x}\,dx$

$$\int_0^\pi \sin rx \ln\frac{x}{\pi-x}\,dx = \int_0^\pi \sin rx \ln x\,dx - \int_0^\pi \sin rx \ln(\pi - x)\,dx =$$

$$(1+\cos r\pi)\int_0^\pi \sin rx \ln x\,dx \tag{B9}$$

which result is also obvious from symmetry considerations. Integrating by parts,

$$\int_0^{\pi} \sin rx \, \ln x \, dx = -\frac{\cos rx \, \ln x}{r}\Big|_0^{\pi} + \frac{1}{r}\int_0^{\pi} \frac{\cos rx}{x} \, dx$$

$$= -\frac{1}{r}\left[\cos r\pi \ln \pi - Ci(r\pi) - \lim_{\varepsilon \to 0} \{\ln \varepsilon - Ci(r\varepsilon)\}\right]$$

(B10)

Hence,

$$\int_0^{\pi} \sin rx \, \ln x \, dx = -\frac{1+\cos r\pi}{r}\left[\cos r\pi \, \ln \pi - Ci(r\pi) + \ln \gamma r\right]$$

$$= \frac{1+\cos r\pi}{r}\left[\ln r\pi\gamma - Ci(r\pi)\right] \quad \text{(B11)}$$

(6) $\int_0^{\pi} \sin rx \left(\frac{1}{x} - \frac{1}{\pi - x}\right) dx$

Again from symmetry considerations,

$$\int_0^{\pi} \sin rx \left(\frac{1}{x} - \frac{1}{\pi - x}\right) dx = (1+\cos r\pi)\int_0^{\pi} \frac{\sin rx}{x} \, dx =$$

$$(1+\cos r\pi) Si(r\pi)$$

(B12)

for $n = m, \quad m \neq 0$

$$J_{C_1}(n,n) = -\frac{1+\cos n\pi}{2n}\left\{Ci(2n\pi) - 2\,Ci(n\pi) + \ln\frac{\gamma n\pi}{2}\right\} \tag{B7}$$

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Hence,

$$\int_0^{\pi} \sin rx \ \ln x \, dx = - \frac{1+\cos r\pi}{r}\Big[\cos r\pi \ \ln \pi - Ci(r\pi) + \ln \gamma r\Big]$$

$$= \frac{1+\cos r\pi}{r}\Big[\ln r\pi\gamma - Ci(r\pi)\Big] \qquad \text{(B11)}$$

(6) $\displaystyle\int_0^{\pi} \sin rx \left(\frac{1}{x} - \frac{1}{\pi - x}\right) dx$

Again from symmetry considerations,

$$\int_0^{\pi} \sin rx \left(\frac{1}{x} - \frac{1}{\pi - x}\right) dx = (1 + \cos r\pi) \int_0^{\pi} \frac{\sin rx}{x} \, dx =$$

$$(1 + \cos r\pi) \, Si(r\pi) \qquad \text{(B12)}$$

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