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LIMIT DESIGN OF A FULL REINFORCEMENT FOR A  
SYMMETRIC CONVEX CUT-OUT IN A UNIFORM SLAB

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Limit Design of a Full Reinforcement for a  
Symmetric Convex Cutout in a Uniform Slab<sup>1</sup>

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Abstract. A recent paper by Weiss, Prager, and Hodge [1]<sup>3</sup> established a design basis for an annular reinforcement of a circular cutout in a uniform slab. In the present paper, the method is extended to deal with a cutout of more arbitrary shape. In addition, the reinforcement is designed so that under a given loading all cross-sections will become fully plastic simultaneously.

1. Introduction. Consider a plane square slab of uniform thickness  $h$ , subject to uniform tensions  $T_x h$ ,  $T_y h$  on its edges. The slab contains a cutout, the shape of which is subjected only to the following limitations: (1) there are at least two perpendicular axes of symmetry; (2) the cutout is convex; (3) the maximum width occurs at an axis of symmetry. The problem is to design the reinforcement of total thickness  $H$  so that the cutout slab will be "safe" under the given loads. Further, the shape of the reinforcement is to be chosen in a particular manner to be defined presently.

This problem is a generalization of the circular cutout considered by Weiss, Prager and Hodge [1]. As such, it is subject

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  3. Numbers in brackets refer to the references at the end of the paper.

to the same limitations of application. In particular, the dimensions of the reinforced part of the slab must be such that it may be reasonably approximated by a curved beam in which shear forces may be neglected.

In a discussion of the paper by Weiss, Prager, and Hodge, English [2] pointed out that if uniaxial loading alone is considered, a non-circular reinforcement could be designed such that two sections become fully plastic simultaneously. In the present paper, this idea will also be extended and a reinforcement designed which becomes fully plastic simultaneously at each section. As will be seen in the later development, this requirement of simultaneous full plasticity does not lead to a unique design, even for a given height so that certain additional conditions may be imposed. Since these conditions are most easily stated in terminology which is yet to be introduced, we shall postpone their discussion to Secs. 3 and 4.

It must, of course, be pointed out that any such analysis will be valid only within the limited framework of beam theory, and must therefore be viewed merely as a first approximation.

2. Method of solution. The method of design used here is based upon a theorem of Prager, Drucker, and Greenberg [3]. This theorem states that if any set of stresses can be found which are in equilibrium with the given loads, and which nowhere violate the yield condition, then the slab will not collapse under the given loads. For the present problem, the unreinforced part of the plate is assumed to be in a state of uniform plane stress, so

that the tractions applied to the edge of the slab will be transmitted directly to the hub. Since the loads cannot cause yielding in the slab, it remains only to consider the state of stress in the hub.

The stress resultants to be considered are defined in Fig. 1. Since shear is to be neglected, the stress resultants consist of an axial force  $N$  and a moment  $M$ . We choose two perpendicular axes of symmetry as the coordinate axes. The equation of the cutout is then given in polar coordinates by

$$r = a(\theta). \quad (2.1)$$

The reinforcement is of radial thickness  $\delta$ , so that the equation of its outer contour is

$$r = a(\theta) + \delta(\theta). \quad (2.2)$$

Vertical equilibrium of the first quadrant demands that

$$N_o = T_y h(a_o + \delta_o). \quad (2.3)$$

However, the moment  $M_o$  at  $\theta = 0$  is indeterminate from static considerations and is temporarily left as a parameter.

Consider now equilibrium of the section OABCD (Figs. 1, 2). As was previously stated, the statically admissible plane stress field in the unreinforced slab is one of uniform stress, so that the external, uniformly distributed loads are transmitted directly to the hub. For convenience of formulation we replace these distributed loads on BC and CD by the equipollent concentrated loads

$$\begin{aligned} F_x &= T_x h(a + \delta) \sin \theta, \\ F_y &= T_y h[(a_o + \delta_o) - (a + \delta) \cos \theta], \end{aligned} \quad (2.4)$$

acting at the midpoints of BC and CD, respectively. Equilibrium in the direction of  $N(\theta)$  then yields

$$\begin{aligned} N(\theta) &= F_x \sin \theta - F_y \cos \theta + N_0 \cos \theta \\ &= h(a + \delta)[T_x \sin^2 \theta + T_y \cos^2 \theta], \end{aligned} \quad (2.5)$$

while moment equilibrium about the midpoint of DE yields

$$\begin{aligned} M(\theta) &= M_0 - N_0[(a_0 + \frac{1}{2}\delta_0) - (a + \frac{1}{2}\delta)\cos \theta] \\ &\quad + F_x[(a + \frac{1}{2}\delta)\sin \theta - \frac{1}{2}(a + \delta)\sin \theta] \\ &\quad + F_y[\frac{1}{2}(a_0 + \delta_0) + \frac{1}{2}(a + \delta)\cos \theta - (a + \frac{1}{2}\delta)\cos \theta] \\ &= M_0 + \frac{1}{2}ah(a + \delta)[T_x \sin^2 \theta + T_y \cos^2 \theta] \\ &\quad - \frac{1}{2}hT_y a_0(a_0 + \delta_0). \end{aligned} \quad (2.6)$$

The interaction formula relating the bending moment and axial force at any cross section is obtained immediately with the aid of Fig. 3 (see also [1]):

$$4s H|M| + N^2 = s^2 H^2 \delta^2. \quad (2.7)$$

If the hub is to become fully plastic everywhere, then Eq. 2.7 must hold for all values of  $\theta$ . Substituting Eqs. 2.5 and 2.6 into 2.7 we obtain an equation for the thickness  $\delta(\theta)$  of the hub. The computations for the case of general loading become quite involved so that we shall consider in detail the two special cases of uniaxial and equal biaxial tensions.

3. Uniaxial tension. Let the direction of the applied tractions be parallel to the x axis, and let the hub be designed so that it restores the cutout slab to full strength. Then  $T_y = 0$

and  $T_x$  is equal to the yield strength  $s$  in simple tension, so that Eqs. 2.5 and 2.6 become

$$\left. \begin{aligned} N(\theta) &= sh(a + \delta)\sin^2 \theta, \\ M(\theta) &= M_0 + \frac{1}{2} sha(a + \delta)\sin^2 \theta. \end{aligned} \right\} \quad (3.1)$$

From the assumed symmetry of the cutout about the  $y$  axis, the axial forces at  $\theta = 0$  and  $\theta = \pi$  must each vanish, since their sum is zero.

We now impose the condition that there must be at least one section where  $M = 0$ . That this is a reasonable requirement follows from the following argument. Since  $M$  is a continuous function of  $\theta$  if it is never zero it is always of the same sign, say positive. However, since the problem is determinate only to within a constant bending moment, the stress resultants obtained by subtracting the minimum value of  $M$  from the bending moment will be statically admissible and nowhere fully plastic. Thus a different reinforcement for full strength could be designed entirely contained within the assumed shape, which is hardly a reasonable basis for design.

In the case considered in [1],  $a$  and  $\delta$  are constant, so that  $M(\theta)$  is an increasing function of  $\theta$  in the first quadrant, and the zero value of  $M(\theta)$  occurs at  $\theta = \pi/2$ . We shall assume that the shape of the cutout and the resulting reinforcement are such that this statement is still valid, i.e., that

$$\frac{d}{d\theta}[a(a + \delta)\sin^2 \theta] \geq 0$$

for  $0 \leq \theta \leq \pi/2$ . In the specific examples considered in Sec. 5 this is always the case.

Returning to Eq. 3.1, we see that since the sections at  $\theta = 0$  and  $\theta = \pi/2$  are fully plastic, the following relations must be valid:

$$\left. \begin{aligned} N(0) &= 0, \\ M(0) &= M_0 = -s H \delta_0^2/4, \end{aligned} \right\} \quad (3.2)$$

and

$$\left. \begin{aligned} N(\pi/2) &= \text{sh}(a_1 + \delta_1) = s H \delta_1, \\ M(\pi/2) &= -s H \delta_0^2/4 + s h a_1 (a_1 + \delta_1)/2 = 0, \end{aligned} \right\} \quad (3.3)$$

where the subscript 1 denotes the value of a quantity at  $\theta = \pi/2$ . Eliminating  $\delta_1$  between Eqs. 3.3, we may solve for the ratio of hub thickness to slab thickness in terms of the hub width at  $\theta = 0$ :

$$\frac{H}{h} = \frac{2a_1^2 + \delta_0^2}{\delta_0^2}. \quad (3.4)$$

The substitution of Eqs. 3.4 and 3.2 into 3.1 then yields

$$\left. \begin{aligned} N(\theta) &= \text{sh}(a + \delta) \sin^2 \theta, \\ M(\theta) &= -\frac{1}{4} \text{sh}[(2a_1^2 + \delta_0^2) - 2a(a + \delta) \sin^2 \theta]. \end{aligned} \right\} \quad (3.5)$$

Since by hypothesis  $M(\theta)$  is increasing in the first quadrant and zero at  $\theta = \pi/2$ , it is everywhere non-positive, so that Eq. 2.9 becomes

$$-\left(\frac{M}{\text{sh}}\right) \left(4 \frac{H}{h}\right) + \left(\frac{N}{\text{sh}}\right)^2 = \left(\frac{H}{h}\right)^2 \delta^2. \quad (3.6)$$

We shall find it convenient to define the following dimensionless quantities:

$$\rho(\theta) = \delta(\theta)/\delta_0, \quad \alpha(\theta) = a(\theta)/\delta_0, \quad \alpha_1 = \alpha(\pi/2). \quad (3.7)$$

Substituting Eqs. 3.4, 3.5, and 3.7 into Eq. 3.6 and simplifying, we obtain

$$\begin{aligned} & [(1 + 2\alpha_1^2)^2 - \sin^4 \theta] \rho^2 + 2\alpha \sin^2 \theta [(1 + 2\alpha_1^2) - \sin^2 \theta] \rho \\ & - [(1 + 2\alpha_1^2)^2 - 2\alpha^2 \sin^2 \theta (1 + 2\alpha_1^2) + \alpha^2 \sin^4 \theta] = 0. \end{aligned} \quad (3.8)$$

Since  $\rho$  must be positive, the larger root of Eq. 3.8 is correct. Using the abbreviation

$$f(\theta) = \sin^2 \theta / (1 + 2\alpha_1^2), \quad (3.9)$$

we may write the solution in the form

$$\rho = -\alpha \frac{f}{1+f} + \frac{\sqrt{1-2\alpha^2 f/(1+f)}}{\sqrt{1-f^2}}. \quad (3.10)$$

With the definitions 3.7 and 3.9, Eq. 3.10 gives the width of the hub at any cross section in terms of the arbitrary width  $\delta_0$  at the section  $\theta = 0$  and the known boundary  $a(\theta)$  of the cutout. The corresponding thickness  $H$  of the hub is then given by Eq. 3.4.

#### 4. Biaxial tension. For full-strength biaxial tension.

$T_x = T_y = s$ , so that Eqs. 2.5 and 2.6 become

$$\left. \begin{aligned} N(\theta) &= sh(a + \delta) \\ M(\theta) &= M_0 + \frac{1}{2} sh[a(a + \delta) - a_0(a_0 + \delta_0)]. \end{aligned} \right\} (4.1)$$

We choose coordinate axes so that the  $x$  axis corresponds to the largest value of  $a$ . As in the case of uniaxial tension, there will be at least one section where  $M = 0$ , i.e., one section under pure tension. Since the average tensile stress will be a maximum across that direction where the cutout radius is a maximum we will

choose this to be the section of pure tension. Thus, setting  $\theta = 0$  in the second equation 4.1 we see that  $M_0 = 0$ . The yield condition 2.9 then states that at  $\theta = 0$

$$N = sh(a_0 + \delta_0) = sH\delta_0$$

so that

$$\frac{H}{h} = \frac{a_0 + \delta_0}{\delta_0} = 1 + \alpha_0. \quad (4.2)$$

The substitution of Eqs. 4.1 and 4.2, together with the definitions 3.7, into the yield condition 2.7 leads to

$$2(1 + \alpha_0) |\alpha(\alpha + \rho) - \alpha_0(1 + \alpha_0)| + (\alpha + \rho)^2 = (1 + \alpha_0)^2 \rho^2. \quad (4.3)$$

If the bending moment at a given section were positive, Eq. 4.3 could be simplified to

$$\alpha_0(2 + \alpha_0)\rho^2 - 2\alpha(2 + \alpha_0)\rho - \alpha^2(3 + 2\alpha_0) + 2\alpha_0(1 + \alpha_0)^2 = 0.$$

However, the discriminant of this equation is

$$\begin{aligned} & 4\alpha^2(2 + \alpha_0)^2 - 4\alpha_0(2 + \alpha_0)[- \alpha^2(3 + 2\alpha_0) + 2\alpha_0(1 + \alpha_0)^2] \\ & = 8(2 + \alpha_0)(1 + \alpha_0)^2(\alpha^2 - \alpha_0^2). \end{aligned}$$

Since, by our choice of axes,  $\alpha < \alpha_0$  this means that the above equation does not have real roots, so that the hypothesis that  $M > 0$  is not a valid one.

Taking  $M < 0$  in Eq. 4.3, solving for  $\rho$  and choosing that root which yields a positive value of  $\rho$ , we obtain

$$\rho = - \frac{\alpha\alpha_0 + (1 + \alpha_0) \sqrt{2\alpha_0(\alpha_0^2 + 2\alpha_0 - \alpha^2)}}{\alpha_0(2 + \alpha_0)}. \quad (4.4)$$

5. Examples. It is interesting to compare the reinforcements here designed with those of Weiss, Prager, and Hodge [1]. For a circular cutout, of unit radius  $\alpha = \text{const.} = 1/\delta_0$ . Eq. 4.4 for biaxial tension then reduces to

$$\rho = \frac{\delta}{\delta_0} = 1, \quad (5.1)$$

i.e., the hub is of constant width  $\delta_0$ . Equation 4.2 then shows that this cutout is precisely that obtained in [1], as would, of course, be predicted by the radial symmetry of both cutout and tractions.

For uniaxial tension, however, the substitution  $\alpha = \text{const.}$  into Eq. 3.10 does not yield any particular simplification. For any given value of  $\delta_0$ , curves may readily be constructed. Fig. 4 shows the case

$$\delta_0 = 1, \quad H = 3.00 \text{ h}, \quad (5.2)$$

Fig. 5 the case

$$\delta_0 = 2, \quad H = 1.50 \text{ h.}$$

and Fig. 6 the case

$$\delta_0 = 3, \quad H = 1.22 \text{ h.} \quad (5.4)$$

In each figure,<sup>4</sup> the dotted line indicates the circular reinforcement computed from the appropriate one of Eqs. 18 and 19 of [1]. Some interpretation of these figures is given in Sec. 6.

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4. The author wishes to thank Mr. E. Levin for carrying out the computations necessary to construct the figures in this section.

As a second example, consider a square cutout of side 2. If the slab is loaded in uniaxial tension perpendicular to a side of the cutout, the equations of the cutout in the first quadrant are

$$\left. \begin{aligned} a(\theta) &= 1/\cos \theta, & 0 \leq \theta \leq \pi/4, \\ a(\theta) &= 1/\sin \theta, & \pi/4 \leq \theta \leq \pi/2 \\ a_0 &= a_1 = 1. \end{aligned} \right\} \quad (5.5)$$

On the other hand, if the uniaxial tension is perpendicular to a diagonal, the x axis must be taken along a diagonal, so that the equation of the first quadrant is

$$\left. \begin{aligned} a(\theta) &= \sqrt{2} / (\sin \theta + \cos \theta), \\ a_0 &= a_1 = \sqrt{2}. \end{aligned} \right\} \quad (5.6)$$

For biaxial tension, the axes must also be chosen along the diagonals, so that  $a(\theta)$  is still given by Eq. 5.6. In Fig. 7 we have sketched the reinforcements for the three types of loading corresponding to a thickness ratio of  $H/h = 1.5$ .

6. Limitations and conclusions. As a preface to any conclusions, the limitations of the results must be pointed out. A detailed discussion of the general validity of beam theory as applied to reinforcement problems is contained in [1] and will not be repeated here. However, it should be pointed out that on the one hand, the depth of the beam must not be too small compared to its length, so that the results pictured in Figs. 5, 6, 7 (b, c) may be only rather crude approximations. On the other hand, the thickness of the hub  $H$  must not be too large

compared to the slab height  $h$  or the question of carrying capacity will arise. This in turn throws some doubt upon Fig. 4, since there  $H/h = 3$ . It follows, therefore, that all results obtained by beam theory must be regarded merely as first approximations.

#### References

- 1 H. J. Weiss, W. Prager, and P. G. Hodge, Jr., Limit design of a full reinforcement for a circular cutout in a uniform slab, J. Appl. Mech. 19, 397-401 (1952).
- 2 J. H. English, discussion of Ref. [1], to be published in J. Appl. Mech.
- 3 D. C. Drucker, W. Prager, and H. J. Greenberg, Extended limit design theorems for continuous media, Q. Appl. Math. 9, 381-389 (1952).

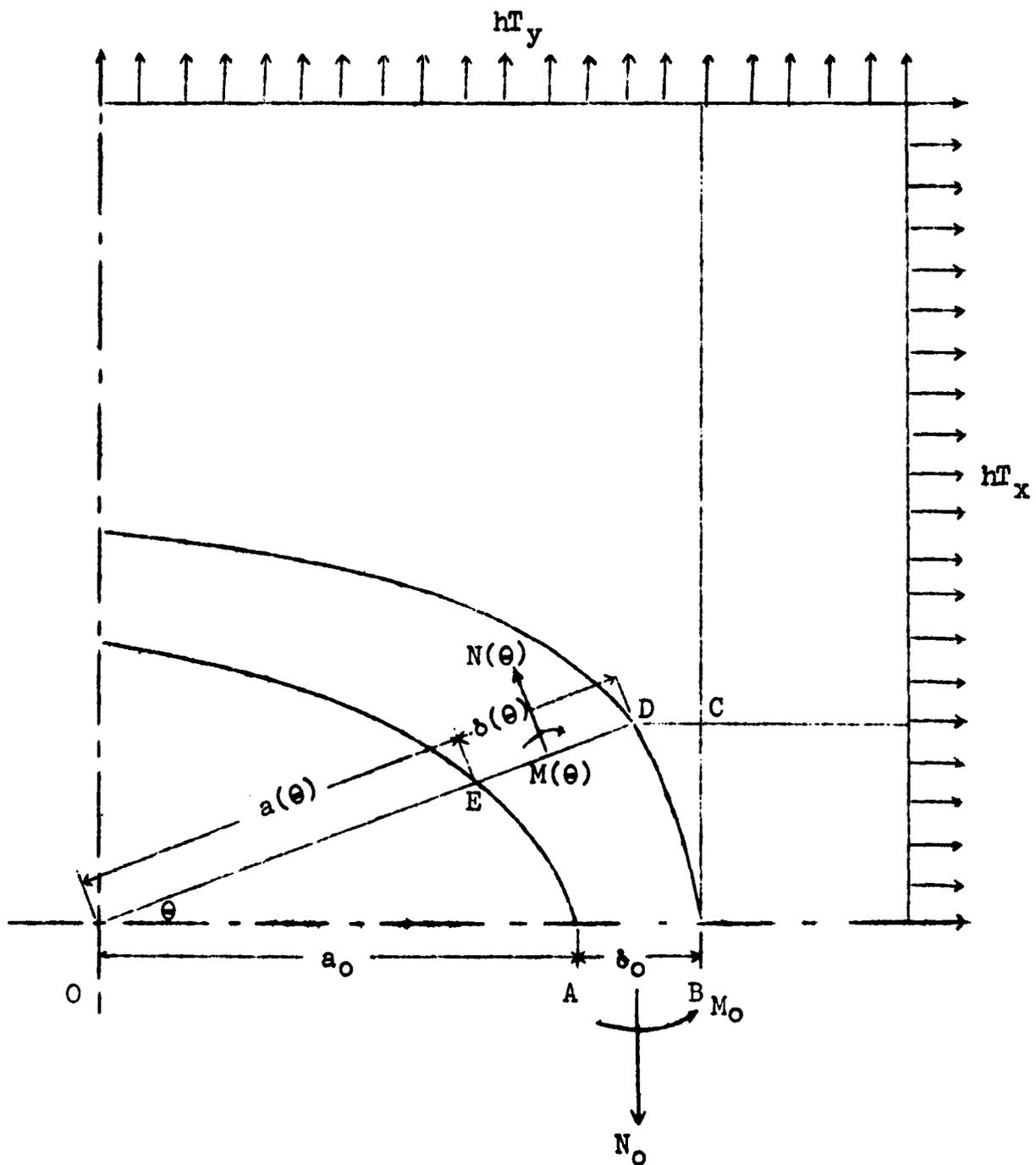


Fig. 1. Stress resultants.

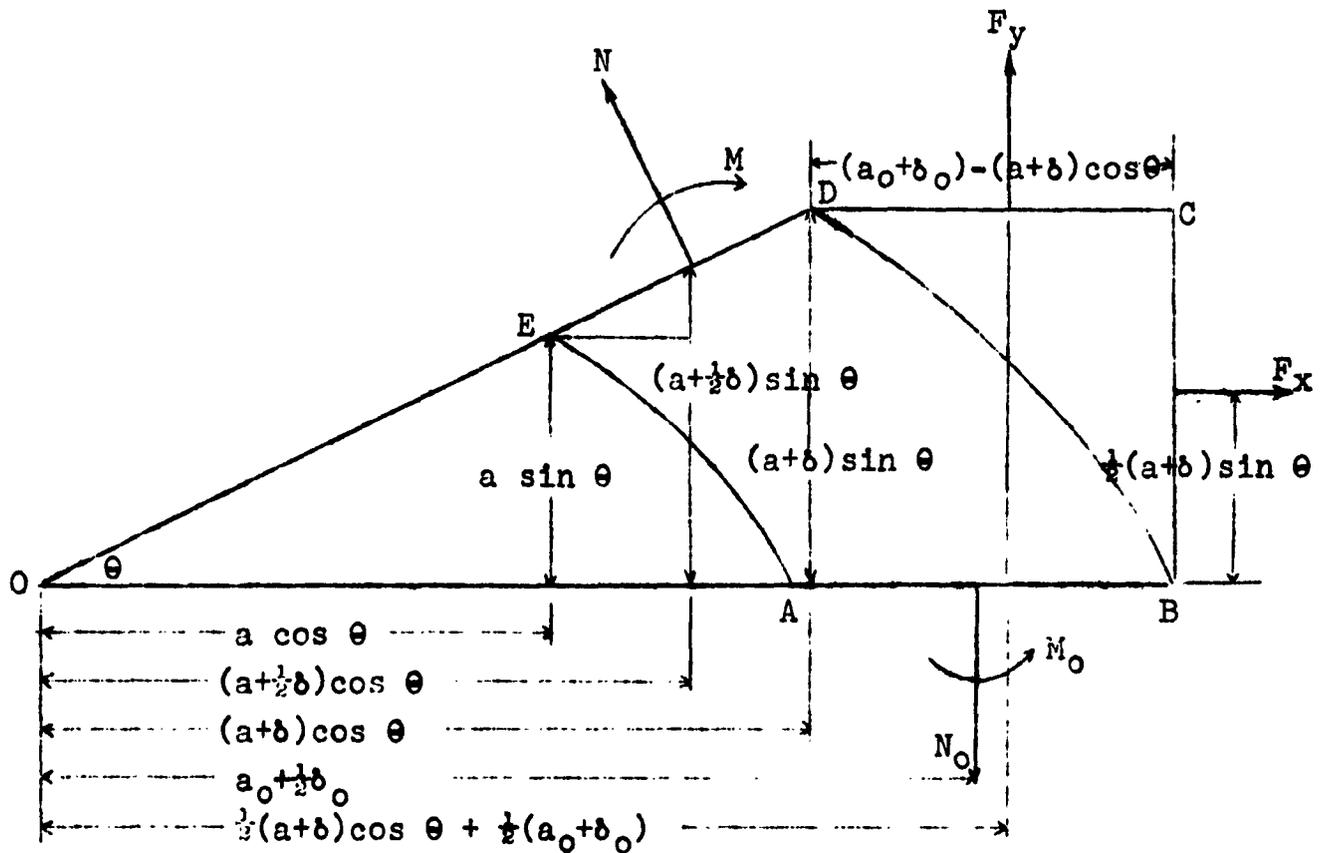


Fig. 2. Equilibrium of portion of hub.

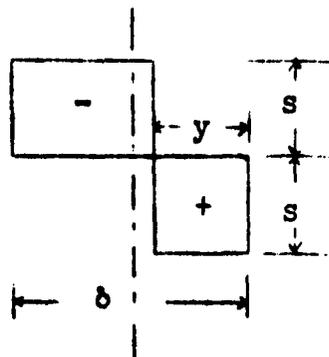


Fig. 3. Fully plastic section of hub.

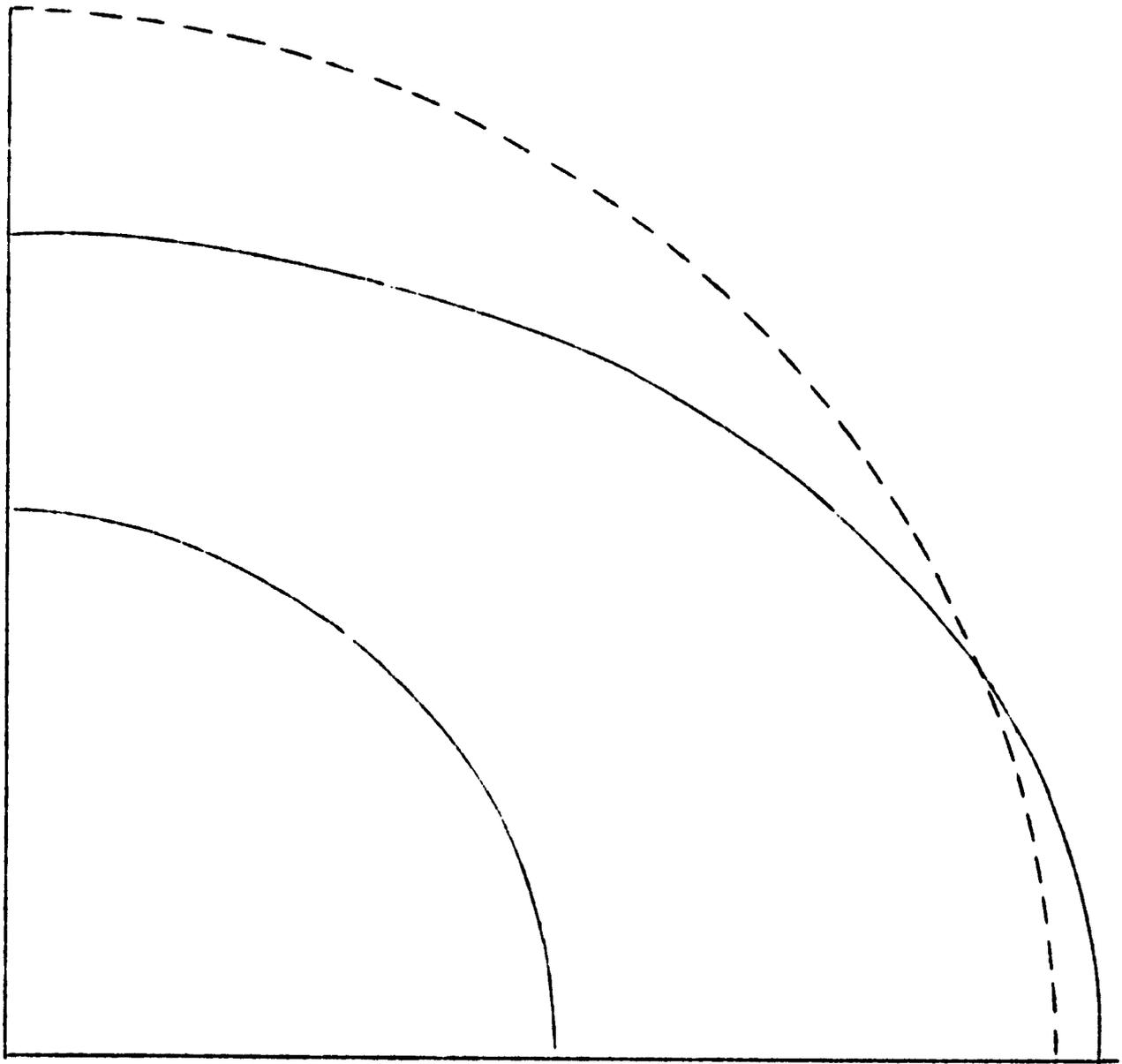


Fig. 4. Full reinforcement for circular cutout,  $\delta_0 = 1$ .

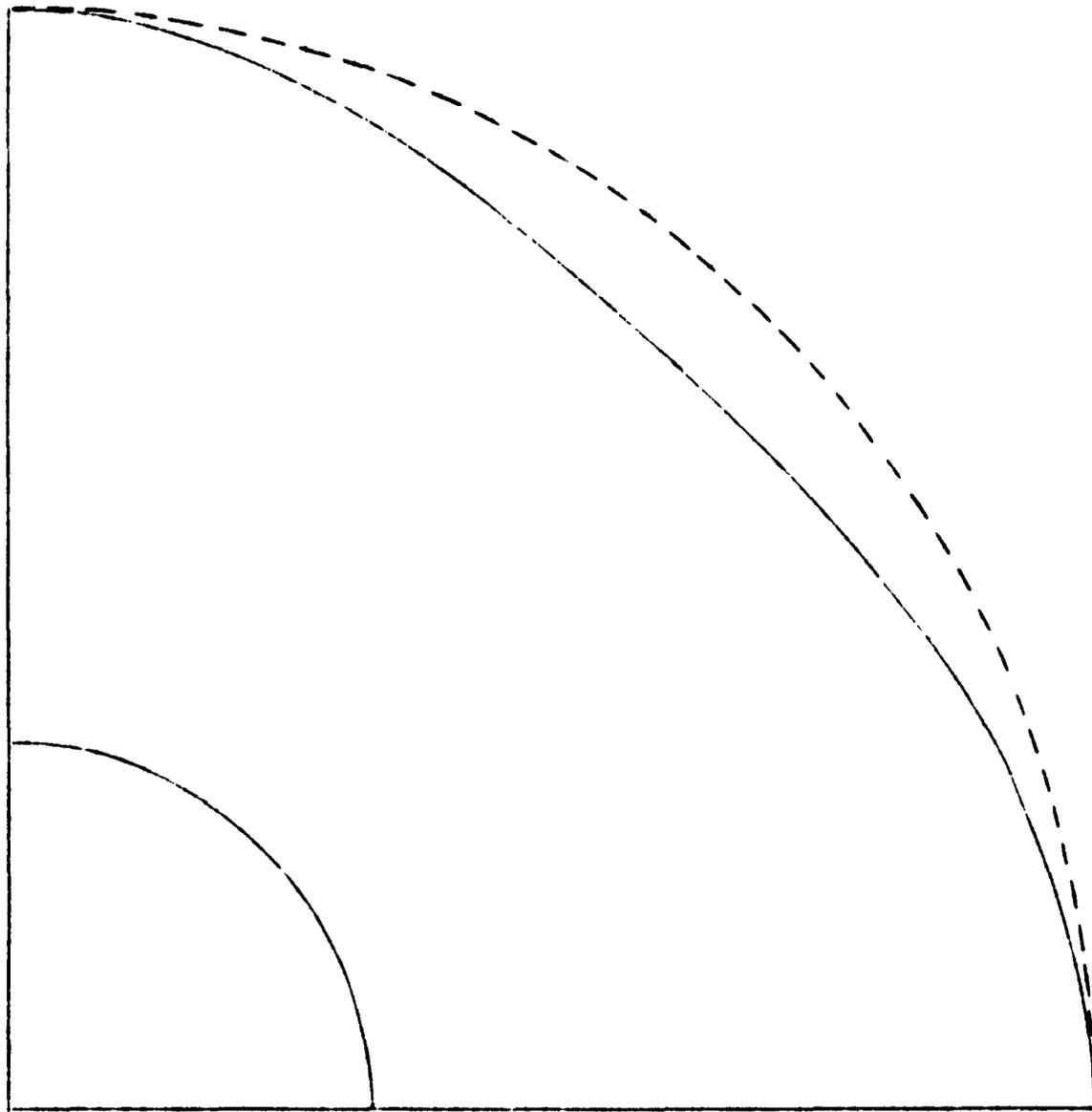


Fig. 5. Full reinforcement for circular cutout,  $\delta_0 = 2$ .

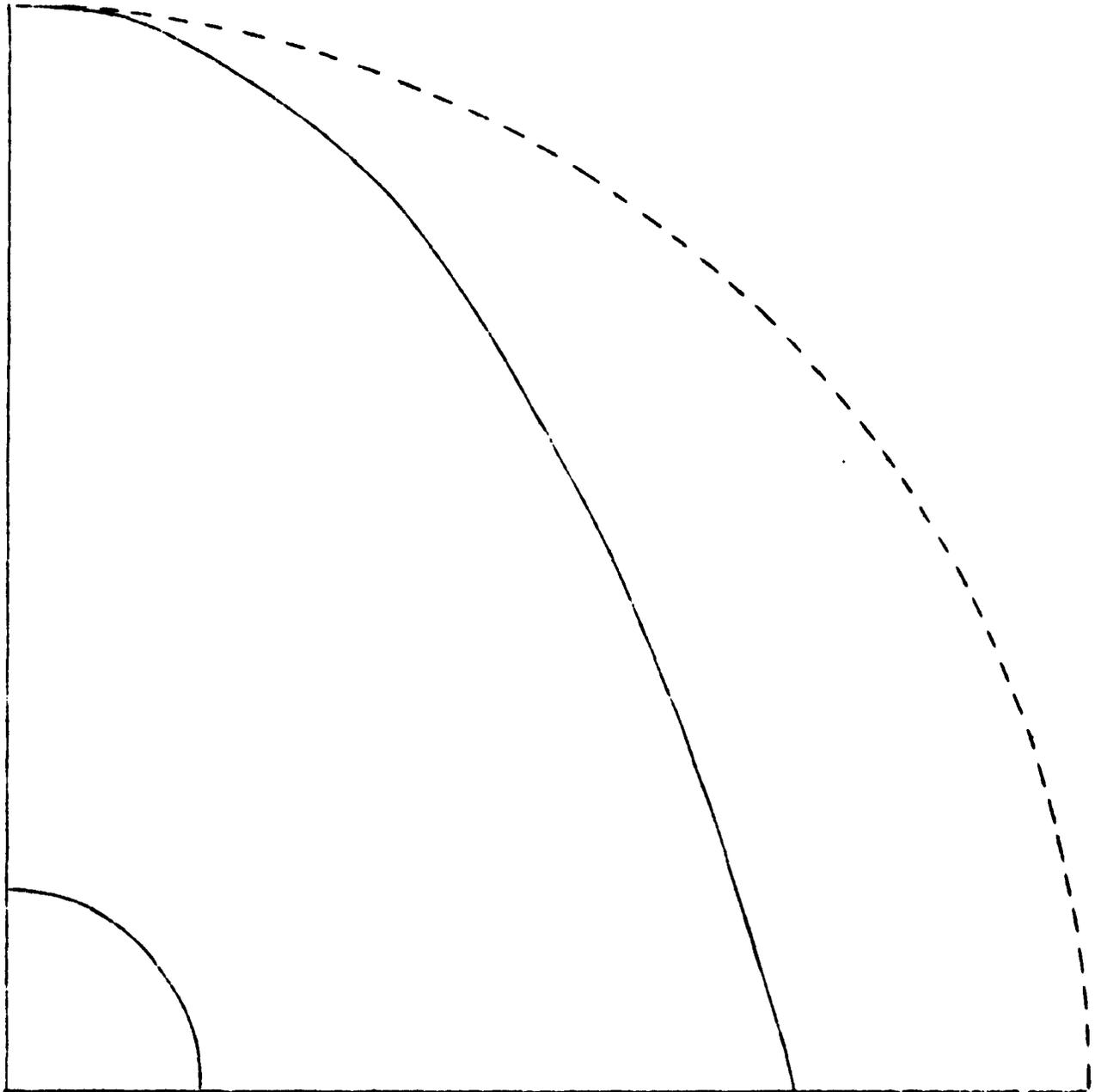
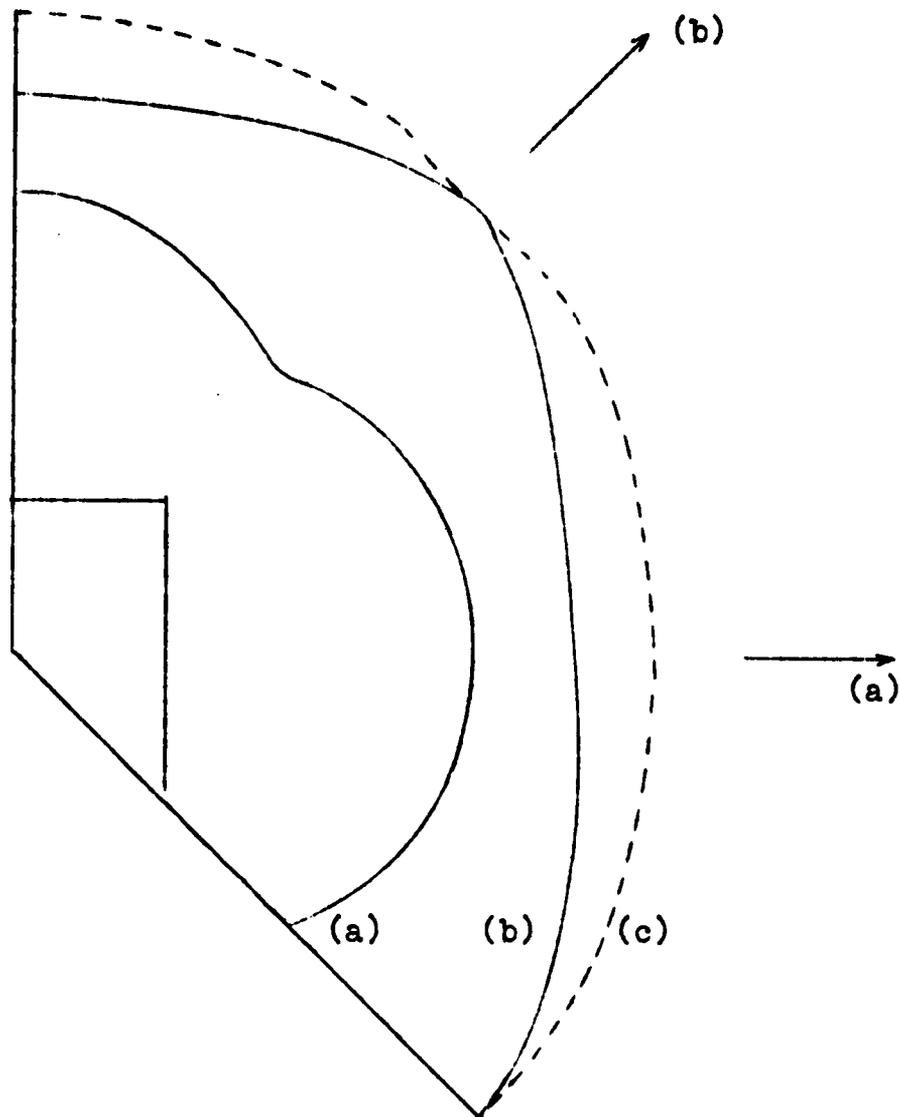


Fig. 6. Full reinforcement for circular cutout,  $\delta_0 = 3$ .



**Fig. 7. Full reinforcements for square cutout.**  
(a) Uniaxial loading perpendicular to side,  
(b) Uniaxial loading along diagonal,  
(c) Biaxial loading.