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DEPARTMENT OF CIVIL ENGINEERING

FORCED MOTIONS OF ELASTIC RODS

by

G. HERRMANN

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FORCED MOTIONS OF ELASTIC RODS

Statement of Problem

An approximate, one-dimensional, theory of vibrations of rods (1)[1], in which the effect of radial inertia and radial shear deformation are taken into account, is contained in the equations of motion

$$a^2 x_1^2 \mu u'' - 8 x_1^2 (\lambda + \mu) u - 4 a x_1^2 \lambda w' + 4 a R = \rho a^2 \ddot{u}$$

$$2 a \lambda u' + a^2 (\lambda + 2\mu) w'' + 2 a Z = \rho a^2 \ddot{w} \qquad [1]$$

the stress-displacement relations

$$2 P_r = 2 P_\theta = x_1^2 [2 a (\lambda + \mu) u + a^2 \lambda w']$$

$$2 P_1 = 2 a \lambda u + a^2 (\lambda + 2\mu) w' \qquad [2]$$

$$4 Q = x^2 a^2 \mu u'$$

the boundary conditions

1. At each point along the length of the bar, one member of each of the products $\dot{u} R$ and $\dot{w} Z$ must be specified.
2. At each end of the bar one member of each of the products $\dot{u} Q$ and $\dot{w} P_1$ must be specified. [3]

and the initial conditions

The initial displacements and velocities must be specified. [4]

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1. Numbers in parentheses refer to the Bibliography at the end of the paper.

In [1] to [4] the symbols have the following meanings:

λ, μ	Lamé's constants of elasticity
ρ	density
u, w	radial and axial displacements, respectively
a	radius of the rod
R, Z	radial and tangential components of traction, respectively, on the cylindrical surface of the rod
χ, χ_1	correction factors
P_r, P_θ, P_z, Q	rod-stress components, defined in (1)

Primes indicate differentiation with respect to z, the coordinate along the axis of the rod, and dots indicate differentiation with respect to time.

The object of this paper is to describe the development of a formal solution of the above equations for a rod of finite length, ℓ, and any admissible combination of time-dependent boundary conditions and arbitrary initial conditions.

The solution will be developed by making use of Lagrange's equations of motion

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} + \frac{\partial V}{\partial q_n} = Q_n \qquad [5]$$

where T is the kinetic energy of the mechanical system considered, V is the potential energy, q_n is the n-th generalized coordinate, $\dot{q}_n$ the n-th generalized velocity, and Q_n the n-th generalized force. As a convenience in specifying the boundary conditions, the following notation will be employed:

$$B_1 = u \quad \text{or} \quad Q \quad \text{at} \quad z = 0$$

$$B_2 = w \quad \text{or} \quad P_z \quad \text{at} \quad z = 0$$

$$B_3 = u \quad \text{or} \quad Q \quad \text{at} \quad z = \ell$$

$$B_4 = w \quad \text{or} \quad P_z \quad \text{at} \quad z = \ell$$

Accordingly, the boundary conditions [3] become

$$B_i = f_i(t) \qquad i = 1, 2, 3, 4 \qquad [3']$$

in which the four functions $f_i(t)$ are prescribed.

The initial conditions [4] are specified by the four arbitrary functions

$$\begin{aligned} u(z, 0) &= u_0(z) \\ w(z, 0) &= w_0(z) \\ \dot{u}(z, 0) &= \dot{u}_0(z) \\ \dot{w}(z, 0) &= \dot{w}_0(z) \end{aligned} \qquad [4']$$

Kinetic Energy

The form of the kinetic energy of the rod may be obtained by performing the appropriate integrations in the expression of the kinetic

energy in terms of the general three-dimensional displacements. Thus, in general, the kinetic energy is

$$T = \frac{\rho}{2} \int_V \left(\dot{u}_r^2 + \dot{u}_\theta^2 + \dot{u}_z^2 \right) d\tau \qquad [6]$$

where u_r, u_θ, u_z are the components of displacement in cylindrical coordinates and the integration is throughout the volume of the rod. With the assumed displacements of the one-dimensional theory (1), which were

$$\begin{aligned} u_r &= \frac{r}{a} u(z,t) \\ u_\theta &= 0 \\ u_z &= w(z,t) \end{aligned} \qquad [7]$$

the kinetic energy of the rod becomes

$$T = \frac{\rho}{2} \int_0^{2\pi} \int_0^a \int_0^l \left(\frac{r^2}{a^2} \dot{u}^2 + \dot{w}^2 \right) r\,dr\,d\theta\,dz \qquad [8]$$

Performing the integrations with respect to θ and r, the volume integral in [8] reduces to a simple integral, extended over the length l of the rod,

$$T = \frac{\pi \rho a^2}{2} \int_0^l \left(\frac{\dot{u}^2}{2} + \dot{w}^2 \right) dz \qquad [9]$$

Potential Energy

The potential energy of the rod is obtained, similarly, by performing appropriate integrations in the general expression for the

potential energy. The latter is

$$V = \int_V W \, d\tau \qquad [10]$$

where the integration is over the volume of the rod and

$$W = \frac{1}{2}\left[\sigma_{rr}\frac{\partial u_r}{\partial r} + \cdots \sigma_{\theta z}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) + \cdots\right] \qquad [11]$$

The assumed displacements [7] are now inserted into [11] and the integrations are performed with respect to r and θ, thus obtaining an expression in terms of u, w, P_r, P_θ, P_z and Q. Use of the stress-displacement relations [2] reduces the result to

$$V = \pi \int_0^\ell \left[2(\lambda+\mu)u^2 + 2\lambda a u w' + \frac{a}{2}(\lambda+2\mu)w'^2 + \frac{\mu x^2 a^2}{4}u'^2\right] dz \qquad [12]$$

For the sake of simplicity, the correction factor x, has been set equal to unity.

Free Vibrations

As a preliminary to the study of forced vibrations, we shall be concerned with the problem of free vibrations, specified by the homogeneous differential equations of motion

$$a^2 x^2 \mu u'' - 8(\lambda+\mu)u - 4a\lambda w' = \rho a^2 \ddot{u}$$
$$2a\lambda u' + a^2(\lambda+2\mu)w'' = \rho a^2 \ddot{w} \qquad [13]$$

and homogeneous boundary conditions

$$B_i = 0 \qquad i = 1, 2, 3, 4 \qquad [14]$$

Considering solutions in the form

$$u(z,t) = U(z)\sin\omega t$$
$$w(z,t) = W(z)\sin\omega t \qquad [15]$$

we may show that the equations of motion [13], together with the boundary conditions [14], are satisfied for an infinite set of discrete frequencies ω_n, each of which corresponds to a mode shape given by functions $U_n(z)$ and $W_n(z)$, which are determined except for a multiplying factor, common to both functions.

The kinetic energy of the rod, vibrating in the n-th principal mode, is, according to [9],

$$T_n(t) = \frac{\omega_n^2 \pi \rho a^2}{2}\int_0^\ell \left(\frac{1}{2}U_n^2 + W_n^2\right) dz \cos^2\omega_n t \qquad [16]$$

and its maximum value is

$$T_{n\,max} = \frac{\omega_n^2 \pi \rho a^2}{2}\int_0^\ell \left(\frac{1}{2}U_n^2 + W_n^2\right) dz \qquad [17]$$

Similarly, the maximum potential energy of the rod is, from [12],

$$V_{n\,max} = \pi\int_0^\ell \Big[2(\lambda+\mu)U_n'^2 + 2\lambda a U_n W_n' + \frac{a^2}{2}(\lambda+2\mu)W_n'^2 + \frac{\mu x^2 a^2}{4}U_n'^2\Big] dz \qquad [18]$$

Employing the principle of conservation of energy, we restate, for future reference, the well known relationship

$$T_{n\,max} = V_{n\,max} \qquad [19]$$

Time-Dependent Boundary Displacements

Whenever a number, m, of time-dependent displacements is specified at the ends of the rod, in a forced vibration problem, it is convenient to solve m static problems in advance; each problem being governed by the equilibrium equations

$$a^2 x^2 \mu u'' - 8(\lambda+\mu) u - 4a\lambda w' = 0$$
$$2a\lambda u' + a^2(\lambda+2\mu) w'' = 0 \qquad [20]$$

The boundary conditions for each static problem are obtained by setting one of the m end displacements equal to unity and letting the remaining three conditions be homogeneous. For example, if the boundary conditions of the forced vibration problem are

$$u(0,t) = f_1(t)$$
$$P_2(0,t) = f_2(t)$$
$$Q(\ell,t) = f_3(t)$$
$$w(\ell,t) = f_4(t)$$

then $m = 2$ and two static problems have to be solved: one with boundary conditions

$$u(0,t) = 1$$
$$P_2(0,t) = 0$$
$$Q(\ell,t) = 0 \qquad [21]$$
$$w(\ell,t) = 0$$

and the other with boundary conditions

$$
\begin{aligned}
u(0,t) &= 0 \\
P_z(0,t) &= 0 \\
Q(\ell,t) &= 0 \\
w(\ell,t) &= 1
\end{aligned}
\qquad [22]
$$

The m static solutions $u(z)$, $w(z)$, which are readily obtained, will be designated $g_{iu}(z)$, $g_{iw}(z)$, where i runs from 1 to m.

General Forced Vibration Problem

The solution of the general forced vibration problem, specified by [1], [2'] and [4'] will be sought in the form

$$
\begin{aligned}
u(z,t) &= \sum_{i=1}^{m} g_{iu}(z)\, f_i(t) + \sum_{n=1}^{\infty} U_n(z)\, q_n(t) \\
w(z,t) &= \sum_{i=1}^{m} g_{iw}(z)\, f_i(t) + \sum_{n=1}^{\infty} W_n(z)\, q_n(t)
\end{aligned}
\qquad [23]
$$

All the functions in [23] have been introduced in preceding sections, and are assumed to have been determined, with the exception of $q_n(t)$, which will be found from Lagrange's equations [5]. It should be noted that [23] satisfies the boundary conditions [3'].

Inserting the forms [23] into the expression [9] for the kinetic energy T, we obtain

$$
T = \frac{\pi \rho a^2}{2} \int_0^{\ell} \left\{ \frac{1}{2} \left[\sum_{i=1}^{m} g_{iu} \dot{f}_i + \sum_{n=1}^{\infty} U_n \dot{q}_n \right]^2 + \left[\sum_{i=1}^{m} g_{iw} \dot{f}_i + \sum_{n=1}^{\infty} W_n \dot{q}_n \right]^2 \right\} dz \qquad [24]
$$

Thus

$$\frac{\partial T}{\partial q_n} = 0 \qquad [25]$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n} = \frac{\pi \rho a^2}{2}\int_0^{\ell}\Big[\sum_{i=1}^{m} g_{iu} U_n \ddot{f}_i + 2\sum_{i=1}^{m} g_{iw} W_n \ddot{f}_i\Big]\,dz$$
$$+ T_{n\,max}\frac{2}{\omega_n^2}\ddot{q}_n \qquad [26]$$

where use has been made of [17] and of

$$\int_0^{\ell}\Big(\frac{1}{2}U_r U_s + W_r W_s\Big)\,dz = 0\,, \qquad \omega_r \neq \omega_s \qquad [27]$$

The relationship [27], which expresses the property of orthogonality of the principal modes of free vibration, can be obtained by substituting the assumed displacements [7] and [15] into Clebsch's theorem (3) and performing the integrations with respect to r and θ. Alternatively, [27] may be obtained from [13] and [14], using Rayleigh's method (4).

Insertion of the displacements [23] into the expression [12] for the potential strain energy V yields

$$V = \pi\int_0^{\ell}\Big\{2(\lambda+\mu)\Big[\sum_{i=1}^{m} g_{iu} f_i + \sum_{n=1}^{\infty} U_n q_n\Big]^2$$
$$+ 2\lambda a\Big[\sum_{i=1}^{m} g_{iu} f_i + \sum_{n=1}^{\infty} U_n q_n\Big]\Big[\sum_{i=1}^{m} g'_{iw} f_i + \sum_{n=1}^{\infty} W'_n q_n\Big]$$
$$+ \frac{a^2}{2}(\lambda+2\mu)\Big[\sum_{i=1}^{m} g'_{iw} f_i + \sum_{n=1}^{\infty} W'_n q_n\Big]^2$$
$$+ \frac{\mu \kappa^2 a^2}{4}\Big[\sum_{i=1}^{m} g'_{iu} f_i + \sum_{n=1}^{\infty} U'_n q_n\Big]^2\Big\}\,dz \qquad [28]$$

Partial differentiation with respect to q_n results in

$$\frac{\partial V}{\partial q_n} = \pi \int_0^{\ell} \Big\{ 4(\lambda+\mu) \sum_{i=1}^{m} g_{iu} f_i U_n$$

$$+ 2\lambda a \sum_{i=1}^{m} \left[g_{iu} f_i W_n' + g_{iw} f_i U_n \right]$$

$$+ a^2 (\lambda + 2\mu) \sum_{i=1}^{m} g_{iw}' f_i W_n' + \frac{\mu \kappa^2 a^2}{2} \sum_{i=1}^{m} g_{iu}' f_i U_n' \Big\} dz$$

$$+ 2 V_{n\,max}\, q_n \qquad [29]$$

where use has been made of [18] and of

$$\int_0^{\ell} \Big[4(\lambda+\mu) U_r U_s + 2\lambda a (U_r W_s' + U_s W_r')$$

$$+ (\lambda + 2\mu) a^2 W_r' W_s' + \frac{\mu \kappa^2 a^2}{2} U_r' U_s' \Big] dz = 0 \qquad \omega_r \neq \omega_s \qquad [30]$$

which states, in another way, the orthogonality property of the principal modes and can be transformed to the relationship [27] by use of the equations of section [13].

Referring to the integral in [29], the primed terms may be integrated by parts. Then, noting that U_n, W_n satisfy [15] and [16], while g_{iu}, g_{iw} satisfy [23] and boundary conditions of the type exemplified by [24], we arrive at

$$\int_0^{\ell} \Big\{ 4(\lambda+\mu) g_{iu} U_n + 2\lambda a (g_{iu} W_n' + g_{iw}' U_n)$$

$$+ (\lambda + 2\mu) a^2 g_{iw}' W_n' + \frac{\mu \kappa^2 a^2}{2} g_{iu}' U_n' \Big\} dz = 0 \qquad [31]$$

Thus, the integral in [29] vanishes and we find, from [29], [31] and [19],

$$\frac{\partial V}{\partial q_n} = 2 q_n V_{n\,max} = 2 q_n T_{n\,max} \qquad [32]$$

The vanishing integral [31] can also be used advantageously to simplify the integral in equation [26]. Assuming, temporarily, the largest possible number of specified, time-dependent displacements at the ends of the rod (m = 4) we obtain, by partial integration,

$$\int_0^{\ell} \Big[\sum_{i=1}^{4} g_{iu} U_n + 2 \sum_{i=1}^{4} g_{iw} W_n \Big] dz$$
$$= \frac{4}{\rho \omega_n^2 a^2} \Big[Q_n(0) + P_{2n}(0) - Q_n(\ell) - P_{2n}(\ell) \Big] \qquad [33]$$

where Q_n and P_{2n} are tractions produced by the displacements of U_n and W_n. Entering this result into [26], we obtain

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n} = \frac{2\pi}{\omega_n^2} \Big[\ddot{f}_1 Q_n(0) + \ddot{f}_2 P_{2n}(0) - \ddot{f}_3 Q_n(\ell) - \ddot{f}_4 P_{2n}(\ell) \Big]$$
$$+ T_{n\,max} \frac{2}{\omega_n^2} \ddot{q}_n \qquad [34]$$

The generalized force Q_n in [5], defined as the coefficient of the increment δq_n of the generalized coordinate q_n, in the expression for the work δW_n, done by the applied physical forces in the displacement produced by δq_n, has to be determined on the basis of this definition. We note, at this point, that not only the q_n's, but also the m functions f_i have to be regarded as generalized coordinates. Consequently, Q_n will not be affected by the presence or absence of any of the functions f_i.

An increment δq_n produces, according to [23] the infinitesimal displacements

$$\delta u = U_n \, \delta q_n$$
$$\delta w = W_n \, \delta q_n \qquad [35]$$

The work δW, done by the applied forces on the surfaces of the rod, has been derived in (1) and is

$$\delta W = 2\pi a \int_0^\ell (R\,\delta u + Z\,\delta w)\, dz$$
$$+ 2\pi \left[Q(\ell)\,\delta u(\ell) + P_z(\ell)\,\delta w(\ell) - Q(0)\,\delta u(0) - P_z(0)\,\delta w(0) \right] \qquad [36]$$

Considering the largest possible number of specified, time-dependent forces, inserting [35] in the above expression and factoring δq_n, we obtain the generalized force $Q_n(t)$

$$Q_n(t) = 2\pi a \int_0^\ell [R U_n + Z W_n]\, dz$$
$$+ 2\pi \left[Q(\ell) U_n(\ell) + P_z(\ell) W_n(\ell) - Q(0) U_n(0) - P_z(0) W_n(0) \right] \qquad [37]$$

Comparing this with [36], Q_n is seen to be the work done by the applied surface tractions in the displacements of the n-th principal mode.

In computing $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n}$, equation [34], we have assumed, as an extreme case, four time-dependent displacements specified at the ends of the rod, ($m = 4$). The generalized force Q_n, equation [37], on the other hand, was derived under the presupposition of four time-dependent tractions acting on the ends of the rod, ($m = 0$). Now, whatever the

value of m, the total number of the terms in the bracket of [34] and in the bracket of [37] will always be four. In view of this, it is convenient to define four quantities H_i, $i = 1, 2, 3, 4$ as follows

$$-H_1 = \frac{2\pi}{\omega_n^2} f_1 Q_n(0) \quad \text{or} \quad 2\pi U_n(0) Q(0), \quad \text{according as}$$

$$B_1 = u \quad \text{or} \quad Q$$

$$-H_2 = \frac{2\pi}{\omega_n^2} f_2 P_{zn}(0) \quad \text{or} \quad 2\pi W_n(0) P_z(0), \quad \text{according as}$$

$$B_2 = w \quad \text{or} \quad P_z$$

$$H_3 = \frac{2\pi}{\omega_n^2} f_3 Q_n(\ell) \quad \text{or} \quad 2\pi U_n(\ell) Q(\ell), \quad \text{according as}$$

$$B_3 = u \quad \text{or} \quad Q$$

$$H_4 = \frac{2\pi}{\omega_n^2} f_4 P_{zn}(\ell) \quad \text{or} \quad 2\pi W_n(\ell) P_z(\ell), \quad \text{according as}$$

$$B_4 = w \quad \text{or} \quad P_z$$

Also we define the quantity

$$Q_n^* = \frac{\omega_n^2}{2T_{n\,max}} \left[2\pi a \int_0^{\ell} (R U_n + Z W_n)\, dz + \sum_{i=1}^{4} H_i \right] \qquad [39]$$

Then Lagrange's equations [5] become

$$\ddot{q}_n + \omega_n^2 q_n = Q_n^* \qquad [40]$$

The general integral of this ordinary differential equation is

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{1}{\omega_n} \int_0^t Q_n^*(\tau) \sin \omega_n (t - \tau)\, d\tau \qquad [41]$$

It remains to determine the constants of integration A_n and B_n, employing the initial conditions [4']. At $t = 0$ we have

$$
\begin{aligned}
u(z,0) &= \sum_{i=1}^{m} g_{iu}(z)\, f_i(0) + \sum_{n=1}^{\infty} A_n U_n \\
w(z,0) &= \sum_{i=1}^{m} g_{iw}(z)\, f_i(0) + \sum_{n=1}^{\infty} A_n W_n \\
\dot{u}(z,0) &= \sum_{i=1}^{m} g_{iu}(z)\, \dot{f}_i(0) + \sum_{n=1}^{\infty} B_n U_n \omega_n \\
\dot{w}(z,0) &= \sum_{i=1}^{m} g_{iw}(z)\, \dot{f}_i(0) + \sum_{n=1}^{\infty} B_n W_n \omega_n
\end{aligned}
\qquad [42]
$$

In view of the orthogonality relationship [27] we multiply the first of equations [42] by $\frac{1}{2} U_s$, the second by W_s, add the two and integrate both sides over the length of the bar.

Solving for A_n we obtain

$$
A_n = \frac{\int_0^{\ell} \left\{ \frac{1}{2}\left[u_0 - \sum_{i=1}^{m} g_{iu}(z)\, f_i(0)\right] U_n + \left[w_0 - \sum_{i=1}^{m} g_{iw}(z)\, f_i(0)\right] W_n \right\} dz}{\int_0^{\ell} \left(\frac{1}{2} U_n^2 + W_n^2\right) dz} \qquad [43a]
$$

B_n is obtained from the third and fourth of equations [41] in an analogous manner

$$
B_n = \frac{\int_0^{\ell} \left\{ \frac{1}{2}\left[\dot{u}_0 - \sum_{i=1}^{m} g_{iu}(z)\, \dot{f}_i(0)\right] U_n + \left[\dot{w}_0 - \sum_{i=1}^{m} g_{iw}(z)\, \dot{f}_i(0)\right] W_n \right\} dz}{\omega_n \int_0^{\ell} \left(\frac{1}{2} U_n^2 + W_n^2\right) dz} \qquad [43b]
$$

This completes the formal solution of the problem.

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Washington 25, D.C. (1)

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RFD 1, Silver Spring, Maryland
Attn: Mechanics Division (2)

Naval Ordnance Test Station
Inyokern, California
Attn: Scientific Officer (1)

NAVY (cont.)

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Attn: Structures Division (1)
Physics Division (1)

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Navy Department
Washington 25, D.C.
Attn: TD-41, Technical Library (1)

Naval Air Experimental Station
Naval Air Materiel Center
Naval Base
Philadelphia 12, Pennsylvania
Attn: Head, Aeronautical Materials Laboratory (1)

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Navy Department
Washington 25, D.C.
Attn: Code P-314 (1)

Officer in Charge
Naval Civil Engr. Research and Eval. Laboratory
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U.S. Naval Proving Grounds
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Air Materiel Command
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Dayton, Ohio
Attn: MCREX-B (E. H. Schwartz) (1)

AIR FORCES (cont.)

Office of Air Research
Wright-Patterson Air Force Base
Dayton, Ohio
Attn: Chief, Applied Mechanics Group (1)

OTHER GOVERNMENT AGENCIES

U.S. Atomic Energy Commission
Division of Research
Washington, D.C. (1)

Argonne National Laboratory
P.O. Box 5207
Chicago 80, Illinois (1)

Director
National Bureau of Standards
Washington, D.C.
Attn: Dr. W. H. Ramberg (1)

U.S. Coast Guard
1300 E Street, N.W.
Washington, D.C.
Attn: Chief, Testing & Development Division (1)

Forest Products Laboratory
Madison, Wisconsin
Attn: L. J. Markwardt (1)

National Advisory Committee for Aeronautics
1724 F Street, N.W.
Washington, D.C. (1)

National Advisory Committee for Aeronautics
Langley Field, Virginia
Attn: Dr. E. Lundquist (1)

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Cleveland Municipal Airport
Cleveland, Ohio
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Washington, D.C.
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Towne School of Engineering
University of Pennsylvania
Philadelphia, Pennsylvania (1)

Professor Lynn Beedle
Fritz Engineering Laboratory
Lehigh University
Bethlehem, Pennsylvania (1)

Professor C. B. Biezeno
Technische Hoogeschool
Nieuw Laan 76
Delft, Holland (1)

Professor M. A. Biot
1619 Broadway
New York, New York (1)

Professor R. L. Bisplinghoff
Massachusetts Institute of Technology
Dept. of Aeronautical Engineering
Cambridge 39, Massachusetts (1)

Dr. Hans H. Bleich
Dept. of Civil Engineering
Columbia University
New York 27, New York (1)

Professor J. A. Bogdanoff
Purdue University
Lafayette, Indiana (1)

Professor B. A. Boley
Dept. of Civil Engineering
Columbia University
New York 27, New York (1)

Professor P. W. Bridgeman
Dept. of Physics
Harvard University
Cambridge, Massachusetts (1)

Professor D. M. Burmister
Dept. of Civil Engineering
Columbia University
New York 27, New York (1)

Dr. V. Cadambe
Assistant Director of the National
Physical Laboratory of India
New Delhi 12, India (1)

Dr. G. F. Carrier
Graduate Division of Applied Mathematics
Brown University
Providence, Rhode Island (1)

Mrs. Hilda L. Cooper
150 Ravine Avenue
Yonkers, New York (1)

Dr. Antoine E. I. Craya
Neyrpic
Boite, Postale 52
Grenoble, France (1)

Professor J. P. Den Hartog
Massachusetts Institute of Technology
Cambridge 39, Massachusetts (1)

Dr. Herbert Deresiewicz
Dept. of Civil Engineering
Columbia University
612 West 116th Street
New York 27, New York (1)

Dr. G. G. Dohrenwend
Rensselaer Polytechnic Institute
Troy, New York (1)

Professor T. J. Dolan
Dept. of Theoretical and Applied
Mechanics
University of Illinois
Urbana, Illinois (1)

Professor Lloyd Donnell
Dept. of Mechanics
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor D. C. Drucker
Brown University
Providence, Rhode Island (1)

Dr. W. Eckert
Watson Scientific Computing Laboratory
612 West 116th Street
New York 27, New York (1)

Dr. H. Ekstein
Armour Research Foundation
Illinois Institute of Technology
Chicago 16, Illinois (1)

Contractors and Other Investigators Actively Engaged in Related Research (cont.)

Engineering Library
Columbia University
New York 27, New York (1)

Professor E. L. Erikson
University of Michigan
Ann Arbor, Michigan (1)

Professor A. C. Eringen
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Dr. W. L. Essmijer
Technische Hoogeschool
Nieuw Laan 76
Delft, Holland (1)

Professor A. M. Freudenthal
Dept. of Civil Engineering
Columbia University
New York 27, New York (1)

Professor B. Fried
Washington State College
Pullman, Washington (1)

Professor K. O. Friedrichs
New York University
Washington Square
New York, New York (1)

Professor M. M. Frocht
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor J. M. Garrelts
Dept. of Civil Engineering
Columbia University
New York 27, New York (1)

Dean J. A. Goff
University of Pennsylvania
Philadelphia, Pennsylvania (1)

Mr. Martin Goland
Midwest Research Institute
4049 Pennsylvania Avenue
Kansas City 2, Missouri (1)

Dr. J. N. Goodier
Dept. of Mechanical Engineering
Stanford University
Stanford, California

Dr. L. E. Goodman
Dept. of Civil Engineering
University of Illinois
Urbana, Illinois (1)

Dr. R. J. Hansen
Massachusetts Institute of Technology
Dept. of Civil & Sanitary Engineering
Cambridge 39, Massachusetts (1)

Professor R. M. Hermes
University of Santa Clara
Santa Clara, California (1)

Professor G. Herrmann
Dept. of Civil Engineering
Columbia University
New York 27, New York (1)

Professor M. Hetényi
Northwestern University
The Technological Institute
Evanston, Illinois (1)

Professor T. J. Higgins
Dept. of Electrical Engineering
University of Wisconsin
Madison 6, Wisconsin (1)

Dr. N. J. Hoff, Head
Dept. of Aeronautical Engineering
& Applied Mechanics
Polytechnic Institute of Brooklyn
99 Livingston Street
Brooklyn 2, New York (1)

Professor M. B. Hogan
University of Utah
Salt Lake City, Utah (1)

Professor D. L. Holl
Iowa State College
Ames, Iowa (1)

Dr. J. H. Hollomon
General Electric Research Labs.
1 River Road
Schenectady, New York (1)

Dr. W. H. Hoppmann
Dept. of Applied Mechanics
John Hopkins University
Baltimore, Maryland (1)

Contractors and Other Investigators Actively Engaged in Related Research (cont.)

Institut de Mathematiqués
Université
post. fah 55
Skopje, Yugoslavia (1)

Professor L. S. Jacobsen
Dept. of Mechanical Engineering
Stanford University
Stanford, California (1)

Professor Bruce G. Johnston
University of Michigan
Ann Arbor, Michigan (1)

Professor K. Klotter
Stanford University
Stanford, California (1)

Professor W. J. Krefeld
Dept. of Civil Engineering
Columbia University
New York 27, New York (1)

Professor B. J. Lazan
Dept. of Materials Engineering
University of Minnesota
Minneapolis, Minnesota (1)

Dr. E. H. Lee
Division of Applied Mathematics
Brown University
Providence, Rhode Island (1)

Professor George Lee
Rensselaer Polytechnic Institute
Troy, New York (1)

Professor J. M. Lessells
Massachusetts Institute of Technology
Cambridge 39, Massachusetts (1)

Library, Engineering Foundation
29 West 39th Street
New York, New York (1)

Professor Paul Lieber
Dept. of Engineering
Rensselaer Polytechnic Institute
Troy, New York (1)

Dr. Hsu Lo
Purdue University
Lafayette, Indiana (1)

Professor C. T. G. Looney
Dept. of Civil Engineering
Yale University
New Haven, Connecticut (1)

Dr. J. L. Lubkin
Midwest Research Institute
4049 Pennsylvania Avenue
Kansas City 2, Missouri (1)

Professor J. F. Ludloff
School of Aeronautics
New York University
New York 53, New York (1)

Professor J. N. Macduff
Rensselaer Polytechnic Institute
Troy, New York (1)

Professor C. W. MacGregor
University of Pennsylvania
Philadelphia, Pennsylvania (1)

Professor Lawrence E. Malvern
Dept. of Mathematics
Carnegie Institute of Technology
Pittsburgh 13, Pennsylvania (1)

Dr. J. H. Marchant
Brown University
Providence, Rhode Island (1)

Professor J. Marin
Pennsylvania State College
State College, Pennsylvania (1)

Dr. W. P. Mason
Bell Telephone Laboratories
Murray Hill, New Jersey (1)

Professor R. D. Mindlin
Dept. of Civil Engineering
Columbia University
632 West 125th Street
New York 27, New York (15)

Dr. A. Nadai
136 Cherry Valley Road
Pittsburgh 21, Pennsylvania (1)

Professor Paul M. Naghdi
Dept. of Engineering Mechanics
University of Michigan
Ann Arbor, Michigan (1)

Contractors and Other Investigators Actively Engaged in Related Research (cont.)

Professor N. M. Newmark
Dept. of Civil Engineering
University of Illinois
Urbana, Illinois (1)

Professor Jesse Ormondroyd
University of Michigan
Ann Arbor, Michigan (1)

Dr. W. Osgood
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Dr. George B. Pegram
Committee on Government Aided Research
313 Low Memorial Library
Columbia University
New York 27, New York (1)

Dr. R. P. Petersen
Director, Applied Physics Division
Sandia Laboratory
Albuquerque, New Mexico (1)

Mr. R. E. Peterson
Westinghouse Research Laboratories
East Pittsburgh, Pennsylvania (1)

Dr. A. Phillips
School of Engineering
Stanford University
Stanford, California (1)

Professor Gerald Pickett
Dept. of Mechanics
University of Wisconsin
Madison 6, Wisconsin (1)

Dr. H. Poritsky
General Electric Research Labs.
Schenectady, New York (1)

Dr. W. Prager
Graduate Division of Applied Mathematics
Brown University
Providence, Rhode Island (1)

RAND Corporation
1500 4th Street
Santa Monica, California
Attn: Dr. D. L. Judd (1)

Dr. S. Raynor
Armour Research Foundation
Illinois Institute of Technology
Chicago 16, Illinois (1)

Professor E. Reissner
Dept. of Mathematics
Massachusetts Institute of Technology
Cambridge 39, Massachusetts (1)

Professor H. Reissner
Polytechnic Institute of Brooklyn
99 Livingston Street
Brooklyn 2, New York (1)

Dr. Kenneth Robinson
National Bureau of Standards
Washington, D.C. (1)

Professor M. A. Sadowsky
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor M. G. Salvadori
Dept. of Civil Engineering
Columbia University
New York 27, New York (1)

Dr. F. S. Shaw
Polytechnic Institute of Brooklyn
99 Livingston Street
Brooklyn 2, New York (1)

Dr. Daniel T. Sigley
Applied Physics Laboratory
John Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland (1)

Dr. C. B. Smith
Department of Mathematics
Walker Hall
University of Florida
Gainesville, Florida (1)

Professor C. R. Soderberg
Massachusetts Institute of Technology
Cambridge 39, Massachusetts (1)

Professor R. V. Southwell
Imperial College of Science and Technology
South Kensington
London S.W. 7, England (1)

Contractors and Other Investigators Actively Engaged in Related Research (cont.)

Professor E. Sternberg
Illinois Institute of Technology
Technology Center
Chicago 16, Illinois (1)

Professor J. J. Stoker
New York University
Washington Square
New York, New York (1)

Mr. R. A. Sykes
Bell Telephone Laboratories
Murray Hill, New Jersey (1)

Professor P. S. Symonds
Brown University
Providence, Rhode Island (1)

Professor J. L. Synge
Dublin Institute for Advanced Studies
School of Theoretical Physics
64-65 Merrion Square
Dublin, Ireland (1)

Professor F. K. Teichmann
Dept. of Aeronautical Engineering
New York University
University Heights, Bronx
New York, New York (1)

Professor S. P. Timoshenko
School of Engineering
Stanford University
Stanford, California (1)

Dr. C. A. Truesdell
Graduate Institute for Applied Mathematics
Indiana University
Bloomington, Indiana (1)

Professor Karl S. Van Dyke
Department of Physics
Scott Laboratory
Wesleyan University
Middletown, Connecticut (1)

Dr. I. Vigness
Naval Research Laboratory
Anacostia Station
Washington, D.C. (1)

Dr. Leonardo Villena
Gran Via J. Antonio 6
Madrid, Spain (1)

Professor E. Volterra
Rensselaer Polytechnic Institute
Troy, New York (1)

Mr. A. H. Wahl
Westinghouse Research Laboratories
East Pittsburgh, Pennsylvania (1)

Professor C. T. Wang
Dept. of Aeronautical Engineering
New York University
University Heights, Bronx
New York, New York (1)

Dr. R. L. Wegel
RFD 2
Peekskill, New York (1)

Professor E. S. Weibel
University of Colorado
Boulder, Colorado (1)

Dr. Alexander Weinstein
Institute of Applied Mathematics
University of Maryland
College Park, Maryland (1)

Professor Dana Young
Yale University
New Haven, Connecticut (1)