AD NUMBER

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ONR ltr, 26 Oct 1977

AD-3584

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SOME RESULTS ON TRUNCATED LIFE TESTS IN THE EXPONENTIAL CASE

BY

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TECHNICAL REPORT NO. 4
FEBRUARY 15, 1953

PREPARED UNDER CONTRACT Nonr-451(00)
(NR-042-017)

FOR

OFFICE OF NAVAL RESEARCH

DEPARTMENT OF MATHEMATICS
WAYNE UNIVERSITY
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SOME RESULTS ON TRUNCATED LIFE TESTS
IN THE EXPONENTIAL CASE

by

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I. Summary

In this report we consider life tests which are truncated as follows. n items are placed on test and it is decided in advance that the experiment will be terminated at min $(X_{r_o,n}, T_o)$, where $X_{r_o,n}$ is a random variable equal to the time at which the r_o 'th failure occurs and T_o is a truncation time, beyond which the experiment will not be run. Both r_o and T_o are assigned before experimentation starts. If the experiment is terminated at $X_{r_o,n}$ (i.e., r_o failures occur before time T_o) then the action in terms of hypothesis testing is the rejection of some specified null-hypothesis. If the experiment is terminated at time T_o (i.e., the r_o 'th failure occurs after time T_o) then the action in terms of hypothesis testing is the acceptance of some specified null-hypothesis. While truncated procedures can be considered for any life distribution, we limit ourselves here to the case where the underlying life distribution is specified by a p.d.f. of the exponential form (1), $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$, $\theta > 0$. Two situations are considered. The first is the non-replacement case where a failure when it occurs during the test is not replaced by a new item. The second is the replacement case where failed items are replaced at once by new items drawn at

(1) The practical justification for using this kind of distribution as a first approximation to a number of test situations is discussed in a recent paper by Davis [1].

[illegible] as the original [illegible] also are given for $E_\theta(r)$, the expected number of observations to come to a decision; $E_\theta(T)$, the expected waiting time for reaching a decision; and $L(\theta)$, the probability of accepting the hypothesis that $\theta = \theta_0$ (θ_0 being the value associated with the null-hypothesis) when θ is the true value. Some procedures are worked out for finding truncated tests meeting specified conditions and a practical illustration is given. Detailed tables will appear elsewhere. Truncated tests considered as special cases of sequential procedures will also be treated in another place.

II. The Derivation of a Truncated Test in the Non-Replacement Case

Let n items drawn at random from a population be placed on life test. Let the underlying p.d.f. of life be of the form $f(x;\theta)$. Items that fail are not replaced and the experiment is truncated at time [(2)] $\min (X_{r_0,n}, T_0)$, where $X_{r_0,n}$ is the time when the r_0'th failure occurs and r_0 and T_0 will be taken as preassigned. T_0 is a truncation time beyond which the experiment does not run. If we define $F(T_0;\theta)$ in the usual way as $F(T_0;\theta) = \int_{-\infty}^{T_0} f(x;\theta)\,dx$, then it follows at once that the probability of reaching a decision requiring exactly k failures is given by

$$(1) \qquad Pr(r=k|\theta) = \binom{n}{k}\left[F(T_0;\theta)\right]^k \left[1-F(T_0;\theta)\right]^{n-k}, \quad k = 1, 2, \ldots, r_0-1$$

and

$$(2) \qquad Pr(r=r_0|\theta) = 1 - \sum_{k=0}^{r_0-1} Pr(r=k/\theta).$$

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(2) For convenience we consider the variate to be time. It is perfectly clear that it can be other things depending on the physical application one is concerned with. Generally the variate (e.g., if it is time) will be non-negative.

Further the expected number of observations to reach a decision is given by

(2) $$E_\theta(r) = \sum_{k=0}^{r_0} k \Pr(r=k|\theta).$$

The formula for $E_\theta(T)$, the expected waiting time for reaching a decision is given by [3]

(4) $$E_\theta(T) = \sum_{k=0}^{n} \Pr{}^*(r=k|\theta)\, E_\theta(T|r=k)$$

$$= T_0\left[\sum_{k=0}^{r_0-1} \Pr{}^*(r=k|\theta)\right] + \sum_{k=r_0}^{n} \Pr{}^*(r=k|\theta) E_\theta(X_{r_0,n}|r=k).$$

In (4), $\Pr^*(r=k|\theta) = \binom{n}{k}\left[F(T_0;\theta)\right]^k\left[1-F(T_0;\theta)\right]^{n-k}$, $k = 0, 1, 2, \ldots, n$.

Suppose the truncation rule is such that a hypothesis H_0 associated with $\theta = \theta_0$ is accepted if $\min(X_{r_0,n}, T_0) = T_0$, i.e., in the particular sample of size n, the waiting time required to observe $X_{r_0,n}$, the r_0'th failure, is more than T_0. Then if $L(\theta)$ is defined as the probability of accepting $\theta = \theta_0$ when θ is true, it follows that

(5) $$L(\theta) = \sum_{k=0}^{r_0-1} \Pr(r=k|\theta).$$

In the special case where the p.d.f. of the life of items is given by $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$, $\theta > 0$, the formulae (1)-(5) become substantially simpler. This is in particular true for $E_\theta(T)$. For the exponential density, (1) and (2) become

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(3) It should be noted that there is an essential difference between $\Pr(r=k|\theta)$ and $\Pr^*(r=k|\theta)$ for $r_0 \leq k \leq n$. $\Pr^*(r=k|\theta)$ is simply the probability that exactly k out of n failures will occur in the interval $(0, T_0)$ while $\Pr(r=k|\theta)$ is the probability that a decision will be reached after exactly k failures are observed. Clearly from the definition of the truncation procedure $\Pr(r=k|\theta) = \Pr^*(r=k|\theta)$ for $0 \leq k \leq r_0-1$. Further $\Pr(r=k|\theta) = 0$ for $r_0 + 1 \leq k \leq n$.

$$(1') \qquad Pr(r=k|\theta) = \binom{n}{k}\left[1 - e^{-T_0/\theta}\right]^k e^{-(n-k)T_0/\theta}, \; k = 0, 1, 2, \ldots, r_0-1$$

and

$$(2') \qquad Pr(r=r_0|\theta) = 1 - \sum_{k=0}^{r_0-1} Pr(r=k|\theta).$$

In the case where the underlying p.d.f. is exponential, (3) becomes

$$(3') \qquad E_\theta(r) = \sum_{k=0}^{r_0-1} k\, b(k;n,p_\theta) + r_0\left[1 - \sum_{k=0}^{r_0-1} b(k;n,p_\theta)\right],$$

where $p_\theta = 1-e^{-T_0/\theta}$ and $b(k;n,p_\theta) = \binom{n}{k} p_\theta^{\,k} (1-p_\theta)^{n-k}$.

It can be readily shown that (3') simplifies to

$$(3'') \qquad E_\theta(r) = np_\theta\left[\sum_{k=0}^{r_0-2} b(k;n-1,p_\theta)\right] + r_0\left[1 - \sum_{k=0}^{r_0-1} b(k;n,p_\theta)\right].$$

This is in a convenient form for calculation. For any preassigned n, T_0, and r_0, $E_\theta(r)$ can be found easily from the Binomial Tables [8] or the Tables of the Incomplete Beta Function [6].

We now derive a simple formula for $E_\theta(T)$ in the exponential case. We first note that for any $f(x;\theta)$, $E_\theta(X_{r_0,n})$, the expected waiting time for the r_0'th failure, is given by

$$(6) \qquad E_\theta(X_{r_0,n}) = \sum_{k=0}^{r_0-1} Pr^*(r=k|\theta)\, E_\theta(X_{r_0,n}|r=k) + \sum_{k=r_0}^{n} Pr^*(r=k|\theta)\, E_\theta(X_{r_0,n}|r=k).$$

Comparing (4) and (6) it is clear that

$$(7) \qquad E_\theta(T) = E_\theta(X_{r_0,n}) + \sum_{k=0}^{r_0-1} Pr(r=k|\theta)\left[T_0 - E_\theta(X_{r_0,n}|r=k)\right].$$

Formula (7) is perfectly general. Let us now make use of the properties of an exponential p.d.f. to simplify (7). This is best done through two lemmas.

Lemma 1: If the underlying p.d.f. of the life distribution is $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}$, $x > 0$, $\theta > 0$, then the conditional expected waiting time for the r_o'th failure in a sample of size n given that exactly k failures, $0 \leq k \leq r_o-1$, have occurred by time T_o is given by

(8) $$E_\theta(X_{r_o,n} \mid r=k) = T_o + E_\theta(X_{r_o-k,n-k}), \quad k = 1, 2, \ldots, r_o-1.$$

In (8), $E_\theta(X_{r_o,n} \mid r=k) \equiv E_\theta(X_{r_o,n} \mid X_{k,n} \leq T_o, X_{k+1,n} \geq T_o)$, $k = 0, 1, 2, \ldots, r_o-1$ is the conditional expected waiting time for which we seek a formula. $E_\theta(X_{r_o-k,n-k})$ is the unconditional expected waiting time to get the (r_o-k)'th failure in a random sample of size $(n-k)$. The proof of Lemma 1 follows directly from results in [3].

Lemma 2: $E_\theta(X_{r_o-k,n-k}) = E_\theta(X_{r_o,n}) - E_\theta(X_{k,n})$, $0 \leq k \leq r_o$

where $E_\theta(X_{0,n})$ is defined as zero for all $n \geq 1$. The proof of Lemma 2 is immediate. In [2] it was shown that

(9) $$E_\theta(X_{r_o,n}) = \theta\left(\frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{n-r_o+1}\right).$$

Thus for any integer k such that $0 \leq k \leq r_o$, it follows (subject to the convention about k = 0) that

(10) $$E_\theta(X_{r_o,n}) - E_\theta(X_{k,n}) = \theta\left(\frac{1}{n-k} + \frac{1}{n-k-1} + \ldots \frac{1}{n-r_o+1}\right) = E_\theta(X_{r_o-k,n-k}).$$

Thus Lemma 2 follows.

Using both Lemmas 1 and 2, (7) becomes

$$(7') \qquad E_\theta(T) = E_\theta(X_{r_o,n}) - \sum_{k=0}^{r_o-1} \Pr(r=k|\theta)\left[E_\theta(X_{r_o,n}) - E_\theta(X_{k,n})\right]$$

$$= \sum_{k=0}^{r_o-1} \Pr(r=k|\theta)E_\theta(X_{k,n}) + \left[1 - \sum_{k=0}^{r_o-1} \Pr(r=k|\theta)\right] E_\theta(X_{r_o,n})$$

$$= \sum_{k=1}^{r_o} \Pr(r=k|\theta)\, E_\theta(X_{k,n}), \text{ since } E_\theta(X_{0,n}) = 0.$$

Thus, in the exponential case, formula (4) for $E_\theta(T)$ simplifies to (7').

We remark paranthetically that for any underlying life distribution (including the exponential), the c.d.f. of the waiting time T, $G_\theta(t)$, is given by

$$(11) \qquad G_\theta(t) = \Pr(T \leq t|\theta) = \Pr(X_{r_o,n} \leq t|\theta), \text{ if } t < T_o$$

$$= 1, \text{ if } t \geq T_o.$$

This result can be useful in finding out more about the waiting time that just its expectation.

In a practical situation one might want a truncated test without replacement which has the following properties:

(i) T_o is preassigned.

(ii) The O.C. curve should be such that $L(\theta_o) \geq 1 - \alpha$ and $L(\theta_1) \leq \beta$. θ_o and θ_1 are preassigned and $\theta_o > \theta_1$.

It is quite easy to accomplish this since conditions (i) and (ii) mean in effect that we are dealing with a binomial situation in which we are testing $p_o = 1 - e^{-T_o/\theta_o}$ against $p_1 = 1 - e^{-T_o/\theta_1}$ with $L(p_o) \geq 1 - \alpha$ and $L(p_1) \leq \beta$. Stated in binomial terms, we are seeking a sample size n and a rejection number r_o such that we will accept the hypothesis that

$p = p_o$ if the number of defectives (failures) in the sample is $\leq r_o - 1$. The hypothesis that $p = p_o$ will be rejected if the number of defectives in the sample of size n is $\geq r_o$. The detailed calculations in any given situation are greatly facilitated by the Binomial Tables [8] or Tables of the Incomplete Beta Function [6]. Tables of practical interest will appear elsewhere.

III. Formulae in the Replacement Case

In this section we assume throughout that the underlying p.d.f. of the life of items is given by $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$, $\theta > 0$. The test is started with n items and any item that fails is replaced at once by a new item drawn at random from the underlying p.d.f. The experiment is truncated at time $\min(X_{r_o,n}, T_o)$, where $X_{r_o,n}$ is the time (measured from the beginning of the entire experiment) when the r_o'th failure occurs and T_o is a truncation time beyond which the experiment does not run. It is then easy to show that the probability of terminating the experiment after exactly k failures have occurred is given by

$$(12) \qquad Pr(r=k|\theta) = \frac{1}{k!} e^{-n T_o/\theta} (n T_o/\theta)^k, \quad k = 0, 1, 2, \ldots, r_{o-1}$$

and

$$(13) \qquad Pr(r = r_o|\theta) = 1 - \sum_{k=0}^{r_o-1} Pr(r=k|\theta).$$

The expected number of observations to reach a decision is given by

$$(14) \qquad E_\theta(r) = \sum_{k=0}^{r_o} k\, Pr(r=k|\theta) = \sum_{k=0}^{r_o-1} k\, p(k;\lambda_\theta) + r_o\left[1 - \sum_{k=0}^{r_o-1} p(k;\lambda_\theta)\right],$$

where $p(k;\lambda_\theta) = \lambda_\theta^k e^{-\lambda_\theta}/k!$ and $\lambda_\theta = n\, T_o/\theta$.

It can be readily shown that (14) simplifies to

$$(15) \qquad E_\theta(r) = \lambda_\theta \left[\sum_{k=0}^{r_o-2} p(k; \lambda_\theta) \right] + r_o \left[1 - \sum_{k=0}^{r_o-1} p(k; \lambda_\theta) \right] .$$

This is in a convenient form for computation. For any preassigned nT_o and r_o, $E_\theta(r)$ can be found easily from Molina's tables on the Poisson distribution [5] or from the tables on the Incomplete Γ-function [7].

The expected waiting time $E_\theta(T)$ is given by a particularly simple formula. Exactly as in the non-replacement case it can be shown that

$$(16) \qquad E_\theta(T) = E_\theta(X_{r_o,n}) + \sum_{k=0}^{r_o-1} Pr(r=k|\theta) \left[T_o - E_\theta(X_{r_o,n}|r=k) \right] .$$

Formula (16) can be simplified by using some properties of the exponential p.d.f. Two lemmas are now stated.

Lemma 3: Let the underlying p.d.f. of the life distribution be $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$ $x > 0$, $\theta > 0$. Let n items be drawn at random from this p.d.f. and put on life test. Let an item that fails be replaced at once by a new item drawn from the underlying p.d.f., then the conditional expected waiting time for the r_o 'th failure (time measured from the beginning of the experiment) given that exactly k failures, $0 \leq k \leq r_o - 1$ have occurred by time T_o is given by

$$(17) \qquad E_\theta(X_{r_o,n} | r=k) = T_o + E_\theta(X_{r_o-k,n}) \quad k = 1, 2, \ldots, r_o-1.$$

This lemma depends on results proved elsewhere [3].

Lemma 4: $E_\theta(X_{r_o-k,n}) = E_\theta(X_{r_o,n}) - E_\theta(X_{k,n}) = \frac{(r_o-k)\theta}{n}$, $0 \leq k \leq r_o$.

This is a consequence of the fact that $E_\theta(X_{s,n}) = s\,\theta/n$ for any integer s.

From Lemmas 3 and 4, (16) becomes

$$(18) \qquad E_\theta(T) = E_\theta(X_{r_o,n}) - \sum_{k=0}^{r_o-1} Pr(r=k|\theta)\left[\frac{(r_o-k)\theta}{n}\right]$$

$$= \frac{\theta}{n}\left[\sum_{k=0}^{r_o-1} k\, Pr(r=k|\theta) + r_o\left(1 - \sum_{k=0}^{r_o-1} Pr(r=k|\theta)\right)\right]$$

$$= \frac{\theta}{n} E_\theta(r).$$

Also in analogy with formula (7') in the non-replacement case, one can write

$$(18') \qquad E_\theta(T) = \sum_{k=1}^{r_o} Pr(r=k|\theta)\, E_\theta(X_{k,n})$$

as the formula which expresses the expected waiting time for the truncated procedure in terms of the unconditional waiting times to get the first r_o failures.

In (7'), $E_\theta(X_{k,n}) = \theta\left(\frac{1}{n} + \frac{1}{n-1} + \dots + \frac{1}{n-k+1}\right)$ (non-replacement).

In (18'), $E_\theta(X_{k,n}) = \frac{k\theta}{n}$ (replacement).

Also $Pr(r=k|\theta)$ is given respectively by (1'), (2') or (12), (13).

Formula (18) bears a strong analogy to Wald's fundamental identity in sequential analysis in which it is shown that under suitable conditions [4], $E(\sum Z_i) = E(Z)\, E(n)$. The Z_i's are identically distributed random variables and n is the smallest integer for which $a < \sum_{i=1}^{n} Z_i < b$ is not satisfied and where a and b are preassigned constants. There is the important difference, however, that in the Wald case information becomes available in discrete amounts, whereas in the life test situation information becomes available continuously. In the Wald case a decision can be made only after some integral number of observations has been taken. In the present case it is possible to stop, however, if it takes too long to make the k'th observation, $0 \leq k \leq r_o$. In a life test, information becomes available in such a way that one does not actually fail a first item, if it takes too long to wait for the first item to fail. More generally, if one has k failures and a decision has not yet been reached to either accept or reject, then one does not actually fail the (k+1)st item if it takes too long to get it. Much more will be said about this question when we treat sequentialized life tests.

In practical situation one might want to find a truncated test with replacement which has the following properties:

(i) T_o is preassigned.

(ii) The O.C. curve should be such that $L(\theta_o) \geq 1 - \alpha$ and $L(\theta_1) \leq \beta$, where θ_o and θ_1 are preassigned and $\theta_o > \theta_1$. Viewed as a test of one Poisson parameter $\lambda_o = 1/\theta_o$ against another $\lambda_1 = 1/\theta_1$ (the λ's are expected number of failures per unit time), this is equivalent to finding a test procedure which will entail observing a Poisson process having average occurrence rate $\lambda_\theta = 1/\theta$, for a length of time equal to nT_o and taking the action of accepting H_o if the number of occurrences observed in the time nT_o is $\leq r_o - 1$ and rejecting H_o if the number of occurrences in the time nT_o is $\geq r_o$.

The details for carrying out the foregoing in any given situation are facilitated by Molina's tables [5] and the tables on the incomplete Γ function [7]. Using these tables, suitable integers n and r_o can always be found so as to make $L(\theta_o) \geq 1 - \alpha$ and $L(\theta_1) \leq \beta$. Several tables dealing with the truncated replacement test have been prepared and will appear elsewhere.

IV. A Test Which Is Not Truncated At A Fixed Time T_o

In [2] it was proved that the "best" region of acceptance for H_o in the Neyman-Pearson sense, in the non-replacement case, for testing a hypothesis H_o that $\theta = \theta_o$ against alternatives of the form $\theta = \theta_1$ $(\theta_o > \theta_1)$ based on the first r out of n ordered observations from an exponential distribution is of the form $\hat{\theta}_{r,n} > C$, where

$$(19) \qquad \hat{\theta}_{r,n} = \frac{x_{1,n} + x_{2,n} + \dots + x_{r,n} + (n-r)x_{r,n}}{r}.$$

Both r and n are preassigned integers. It is easily verified that

$$(20) \quad x_{1,n}+x_{2,n}+\dots+x_{r,n}+(n-r)x_{r,n}=nx_{1,n}+(n-1)\ (x_{2,n}-x_{1,n})+\dots+(n-r+1)\ (x_{r,n}-x_{r-1,n}).$$

Introducing new random variables ξ_i defined by

$$(21) \qquad \xi_1 = nx_{1,n}$$

$$\xi_i = (n-i+1)\ (x_{i,n} - x_{i-1,n}), \qquad i = 2, 3, \dots, r$$

it is clear that $\theta_{r,n} > C$ can be rewritten as

$$(22) \qquad \xi_1 + \xi_2 + \dots + \xi_r > rC.$$

It is now asserted that (22) carries with it the implication that the test is truncated. This is evident since the ξ_i's are positive random variables which are monotonically non-decreasing as time goes on. More precisely the experiment will be truncated at time t_1 (with acceptance of H_o) if no failures have occurred by time $t_1 = r\,C/n$. More generally, suppose that i failures $(1 \leq i \leq r-1)$ have occurred, without reaching a decision; i.e., suppose that $\sum_{k=1}^{i} \xi_k < rC$, then the experiment can be truncated before the (i+1)st failure occurs, if the time t_{i+1} between the i th and (i + 1)st failure is such that

$$(23) \qquad t_{i+1} > t(\xi_1, \xi_2, \ldots, \xi_k) = \frac{rC - \sum_{k=1}^{i} \xi_k}{n-i}.$$

Truncation would occur after an additional waiting time of t^* beyond the time $x_{i,n}$ (when the i th failure occurs). If none of the inequalities (23) are true for any $0 \leq i \leq r-1$ (define $\sum_{k=1}^{i} \xi_k = 0$, if $i = 0$) then the experiment will be terminated after the occurrence of the first r failures In this case

$$(24) \qquad \sum_{i=1}^{r} \xi_i < rC.$$

The action taken on the basis of (24) is the rejection of H_0, and the total time required before taking this action would be $x_{r,n}$.

We now proceed to derive certain properties of the test based on $\hat{\theta}_{r,n}$. To do so we recall, [2], that the ξ_i are independent random variables, each of which is distributed with the same p.d.f. $\frac{1}{\theta} e^{-x/\theta}$, $x > 0$, $\theta > 0$. Thus the problem has been reduced to one to which the theory of section 3 can be applied. From the theory developed there it follows at once that

$$(25) \qquad Pr(\rho = k \mid \theta) = p(k; \mu_\theta), \; k = 0, 1, 2, \ldots, r-1$$

and

$$(26) \qquad Pr(\rho = r \mid \theta) = 1 - \sum_{k=0}^{r-1} p(k; \mu_\theta).$$

In (25) and (26) $\mu_\theta = r\,C/\theta$ and $p(k; \mu_\theta) = \mu_\theta^{\,k} e^{-\mu_\theta}/k!$

Thus in analogy with (14) and (15) we have

$$(27) \qquad E_\theta(\rho) = \sum_{k=0}^{r} k\, Pr(\rho = k \mid \theta) = \mu_\theta \left[\sum_{k=0}^{r-2} p(k; \mu_\theta)\right] + r\left[1 - \sum_{k=0}^{r-1} p(k; \mu_\theta)\right].$$

Further $E_\theta(T)$, the expected waiting time to reach a decision, can be written as

(28) $$E_\theta(T) = \sum_{k=1}^{r} \Pr(\rho = k \mid \theta)\, E_\theta(X_{k,n})$$

where $\Pr(\rho = k \mid \theta)$ is given by (25) and (26) and $E_\theta(X_{k,n})$ is given by (9) with r_o replaced by k.

Finally $L(\theta)$, the probability of accepting $\theta = \theta_o$ when θ is true is given by $L(\theta) = \sum_{k=0}^{r-1} p(k; \mu_\theta)$.

Up to this point in the present section we have been treating the non-replacement situation. It is interesting to see what happens if failed items are replaced at once by new items drawn from the p.d.f. $\frac{1}{\theta} e^{-x/\theta}$. As in Section 3, let $x_{k,n}$ be the time when the k th failure occur (whether it be an original item or replacement item) measured from the beginning of the experiment. It can be shown, in the replacement case, that if one starts with n items, then the "best" region of acceptance for H_o in the Neyman-Pearson sense for testing a hypothesis H_o that $\theta = \theta_o$ against alternatives of the form $\theta = \theta_1$ $(\theta_1 > \theta_o)$ based on the first r failure times $x_{1,n}, x_{2,n}, \ldots, x_{r,n}$ is of the form $\hat{\theta}_{r,n} > C$, where $\hat{\theta}_{r,n}$ is now simply equal to

(29) $$\hat{\theta}_{r,n} = n\, x_{r,n}/r.$$

Thus the region of acceptance for H_o is of the form $x_{r,n} > C^*$. But this means, of course, that we are dealing with a truncated test. $x_{r,n} > C^*$ as a region of acceptance means in words that the test is terminated at min $(X_{r,n}, C^*)$ with acceptance of H_o if truncation occurs at C^* and rejection of H_o if truncation occurs at $X_{r,n}$. Thus the theory of Section 3 is completely applicable.

V. A Numerical Example

As an illustration of the theory we consider three test procedures which have essentially the same O.C. curve. Specifically, it is assumed that

that the underlying p.d.f. is $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$, $\theta > 0$. We wish to test H_o: $\theta_o = 1500$ hours against H_1: $\theta_1 = 500$ hours, with $\alpha = \beta = .05$; i.e., we want $L(\theta_o) = 1 - \alpha = .95$ and $L(\theta_1) = \beta = .05$. Due to the fact that we are limiting discussion to non-randomized tests, we will be satisfied with a test procedure for which $L(\theta_o) = .95$ and $L(\theta_1) \leq .05$.

Using the results in [2] and in the earlier part of this report it can be readily verified that the following three tests have virtually the same O.C. curve:

(A) 20 items are taken at random from the lot and placed on test. Items which fail are not replaced. At each moment t, compute $\sum_{k=1}^{i} (n-k+1)x_{k,n} + (n-i)(t-x_{i,n})$, where i is the number of failures which have occurred before time t. If this sum exceeds 8150 for any i such that $0 \leq i \leq 9$, stop the experiment at time t and accept H_o. If 10 failures occur and this sum is less than 8150, then reject H_o (accept H_1). As pointed out in Section 4 this test is equivalent to accepting H_o if $\hat{\theta}_{10,20} > 815$ and rejecting H_o if $\hat{\theta}_{10,20} < 815$.

(B) 20 items are taken at random from the lot and placed on test. Failed items are not replaced. If $\min[X_{10,20}, 540] = 540$ truncate the experiment at time 540 hours with the acceptance of H_o. If $\min[X_{10,20}, 540] = X_{10,20}$ truncate the experiment at $X_{10,20}$ with the rejection of H_o.

(C) 20 items are taken at random from the lot and placed on test. An item which fails is replaced at once by a new item from the original lot. The time $X_{i,n}$ when the i th failure occurs is measured from the beginning of the experiment. If $\min[X_{10,20}, 407.5] = 407.5$, truncate the experiment at 407.5 hours with the acceptance of H_o. If $\min[X_{10,20}, 407.5] = X_{10,20}$, truncate the experiment at $X_{10,20}$.

It follows from Sections 3 and 4 that the O.C. curves and the probability of terminating experimentation with exactly $r = k$ observations $(0 \leq k \leq 10)$ and hence (in particular) $E_\theta(r)$ are exactly the same for procedures A and C. In Table 1 we give $L(\theta)$, $E_\theta(r)$, and $E_\theta(T)$ for the tests A, B, C at selected values of θ.

Table 1

Properties of Three Test Procedures

	$L(\theta)$			$E_\theta(r)$			$E_\theta(T)$		
Mean Life θ	A	B	C	A	B	C	A	B	C
250	.000	.000	.000	10	10	10	167.2	167.2	125.0
500	.038	.043	.038	9.93	9.94	9.93	331.4	331.6	248.3
750	.355	.366	.355	9.10	9.25	9.10	444.7	453.5	341.3
1000	.698	.702	.698	7.68	8.06	7.68	481.8	509.1	384.0
1250	.876	.877	.876	6.39	6.93	6.39	484.8	529.2	399.3
1500	.950	.950	.950	5.39	6.02	5.39	474.7	536.0	404.5
1750	.979	.979	.979	4.64	5.30	4.64	466.0	538.3	406.3
2000	.991	.991	.991	4.07	4.73	4.07	458.3	539.4	407.0
2250	.996	.996	.996	3.62	4.27	3.62	452.3	539.7	407.3
2500	.998	.998	.998	3.26	3.88	3.26	447.3	539.9	407.4

It is easy to verify that for all three procedures $\lim_{\theta \to 0} E_\theta(r) = 10$ and $\lim_{\theta \to \infty} E_\theta(r) = 0$. Further for procedures A and B, $\lim_{\theta \to 0} E_\theta(T)/E_\theta(X_{10,20}) = 1$. Hence $E_\theta(T) \sim .66877\theta$ as $\theta \to 0$. For procedure C, $\lim_{\theta \to 0} E_\theta(T)/\theta = 1/2$. As $\theta \to \infty$, $\lim_{\theta \to \infty} E_\theta(T) = 407.5$ in procedures A and C and $= 540$ in procedure B.

It is quite clear that Procedures A, B, and C are not optimum in several respects. It should be possible to truncate with a smaller $E_\theta(r)$ as θ gets small and with a smaller $E_\theta(T)$ as θ gets large. We shall study this question in a report dealing with sequentialized life tests. In any event we can see already that by taking advantage of the fact that failures are ordered in time, it will be possible:

(1) to come to a decision (rejection of H_o) after a short waiting time (small $E_\theta(T)$) if the mean life is low and

(2) to come to a decision (acceptance of H_o) after a small number of failures (small $E_\theta(r)$) if the mean life is large. If the mean life $\longrightarrow \infty$, it will be possible to stop (with acceptance of H_o) at some time T^* without any failures at all occurring.

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