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ON SLOW VISCO-PLASTIC FLOW

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1. Introduction. The incompressible visco-plastic material considered in the following is a Bingham solid [1].** The mechanical behavior of this solid is most readily visualized in the following manner. Let a Newtonian viscous liquid, a perfectly plastic Mises solid [2], and a visco-plastic Bingham solid be subjected to the same velocity strain. The stress in the Bingham solid is then obtained by adding the stresses in the Newtonian liquid and the Mises solid.

Whereas specific boundary value problems concerning the slow flow of a Bingham solid have been discussed in the literature (see, for instance, [3]), few general results concerning this type of visco-plastic flow seem as yet to be available. In the present paper a uniqueness theorem and two extremum principles are established.

2. Notation. Latin subscripts take the range 1, 2, 3, and the summation convention operates on repeated subscripts. The coordinates x_i are rectangular and Cartesian, and differentiation with respect to x_i is indicated by a comma subscript followed by the subscript i ($f_{,i} = \partial f/\partial x_i$).

The following boundary value problem will be considered. The incompressible Bingham solid under consideration occupies the three-dimensional region V which is bounded by the surface S. The body force F_i (per unit of volume) is given throughout V. The surface

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** Numbers in square brackets refer to the bibliography at the end of the paper.

traction T_i is prescribed on the portion S_T of S, and the velocity v_i on the remainder S_V. If $S_T \equiv S$, the given surface tractions must, of course, satisfy the conditions of equilibrium, $\int T_i dS + \int F_i dV = 0$, and if $S_V \equiv S$, the given velocities must satisfy the condition of incompressibility, $\int v_i n_i dS = 0$, where n_i is the unit exterior normal of S. From the data on the surface, the velocity field $v_i(x)$ and the stress field $\sigma_{ij}(x)$ are to be determined throughout V under the assumption that the effects of inertia are negligible when compared to the effects of the viscous stresses and yield stresses.

It is convenient to write the stress tensor in the form

$$\sigma_{ij} = - p\delta_{ij} + s_{ij}, \tag{1}$$

where $p = - \sigma_{kk}/3$ is the mean pressure, and s_{ij} the stress deviation. The velocity strain is defined by

$$\varepsilon_{ij} = \frac{1}{2} (v_{i,j} + v_{j,i}). \tag{2}$$

The incompressibility of the solid requires that

$$\varepsilon_{ii} = v_{i,i} = 0. \tag{3}$$

The following positive in variants of velocity strain and stress deviation prove useful in the analytical description of the mechanical behavior of the solid:

$$I = (2\varepsilon_{ij}\, \varepsilon_{ij})^{1/2}, \qquad J = (\tfrac{1}{2}\, s_{ij}s_{ij})^{1/2}. \tag{4}$$

3. Basic relations. The general relations between the stress deviation and the velocity strain in an incompressible Bingham solid were first given by Hohenemser and Prager [4], [5]. In the present notation, these relations have the form

$$2\mu\varepsilon_{ij} = \begin{cases} 0 & \text{if } J < k, \\ (1 - \frac{k}{J})\, s_{ij} & \text{if } J \geq k. \end{cases} \tag{5}$$

By squaring (5) and using (4), we find that

$$\mu I = \begin{cases} 0 & \text{if } J < k, \\ J - k & \text{if } J \geq k. \end{cases} \tag{6}$$

Substitution of (6) into the second Eq. (5) yields

$$s_{ij} = 2(\mu + \frac{k}{I})\varepsilon_{ij} \text{ if } I \neq 0. \tag{7}$$

Of course, whenever the stress remains below the yield limit, the velocity strain vanishes, and the stress cannot be expressed in terms of the velocity strain.

Since the inertia effects are neglected, the stresses must satisfy the equations of equilibrium; these are

$$\sigma_{ij,j} + F_i = 0 \text{ in } V, \tag{8}$$

and

$$\sigma_{ij}n_j = T_i \qquad \text{on } S_T. \tag{9}$$

Introducing (1) into (8) and eliminating p by cross differentiation we obtain

$$s_{ij,jk} - s_{kj,ji} + F_{i,k} - F_{k,i} = 0 \tag{10}$$

as the condition of equilibrium for the stress deviation.

In the following, the principle of virtual work will be used repeatedly. Let v_i'' be a continuous velocity field with piecewise continuous first derivations satisfying (3) and σ_{ij}' an entirely unrelated continuous stress field with piecewise continuous first derivatives satisfying (8). Furthermore, set $\varepsilon_{ij}'' = \frac{1}{2}(v_{i,j}'' + v_{j,i}'')$ and $T_i' = \sigma_{ij}'n_j$. The principle of virtual work is then expressed by the equation

$$\int \sigma_{ij}' \; \varepsilon_{ij}'' \, dV = \int T_i' \; v_i'' \, dS + \int F_i \; v_i'' \, dV. \tag{11}$$

4. Uniqueness. A stress field σ_{ij}, a velocity field v_i, and the associated field of velocity strain $\varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ will be said to constitute a solution of the boundary value problem formulated in Sec. 2 if they satisfy the boundary conditions and Eqs. (3), (5), and (8).

Let us assume that the boundary value problem admits a second solution σ'_{ij}, v'_i, ε'_{ij}. Since each solution corresponds to the same body forces F_i, the principle of virtual work yields

$$\int (\sigma_{ij} - \sigma'_{ij})(\varepsilon_{ij} - \varepsilon'_{ij})dV = \int (T_i - T'_i)(v_i - v'_i)dS. \quad (12)$$

The right-hand side of (12) vanishes because the first factor of the integrand vanishes on S_T and the second factor on S_V. To prove that our boundary value problem defines a unique field of velocity strain, we shall prove that the integral on the left-hand side of (12) is positive unless $\varepsilon_{ij} = \varepsilon'_{ij}$ throughout V.

On account of the incompressibility of the solid, the integrand on the left-hand side of (12) can be written as $A = (s_{ij}-s'_{ij}) \cdot (\varepsilon_{ij} - \varepsilon'_{ij})$. In discussing the sign of this expression, we must consider four types of region.

(_i_) Where neither ε_{ij} nor ε'_{ij} vanishes, we have by (7) and (4),

$$A = 2\mu(\varepsilon_{ij} - \varepsilon'_{ij})(\varepsilon_{ij} - \varepsilon'_{ij}) + \frac{k}{II'}(I + I')(II' - 2\varepsilon_{ij}\varepsilon'_{ij}). \quad (13)$$

The first expression on the right-hand side of (13) is positive unless $\varepsilon_{ij} = \varepsilon'_{ij}$, and the second expression is non-negative by the Schwarzian inequality. Thus, A can vanish in this type of region only if $\varepsilon_{ij} = \varepsilon'_{ij}$.

(_ii_) Where $\varepsilon_{ij} \neq 0$ but $\varepsilon'_{ij} = 0$, we have, according to (5)

$$J > k, \quad J' \leq k \quad (14)$$

and hence

$$A = (s_{ij} - s'_{ij})\varepsilon_{ij} = \frac{1}{2\mu}\frac{J - k}{J}[s_{ij}s_{ij} - s_{ij}s'_{ij}]. \quad (15)$$

Here, the factors in front of the bracket are positive. Moreover,

$$s_{ij}s_{ij} = 2J^2 > 2JJ'$$

by (14), and hence

$$s_{ij}s_{ij} - s_{ij}s'_{ij} > 2JJ' - s_{ij}s'_{ij} \geq 0 \quad (16)$$

by the Schwarzian inequality. Thus, A cannot vanish in this type of region.

(iii) It can be shown in a similar manner that A cannot vanish where $\varepsilon_{ij} = 0$ but $\varepsilon'_{ij} \neq 0$.

(iv) Where $\varepsilon_{ij} = \varepsilon'_{ij} = 0$, we have $A = 0$.

It follows from the preceding discussion that the left-hand integral in (12) can vanish, as it must do, only if $\varepsilon_{ij} = \varepsilon'_{ij}$ throughout V. The field of velocity strain is therefore seen to be unique. Since this field determines the velocity field to within a rigid body motion, the velocity field will also be unique except when $S_V = 0$. In this case, the velocity field is determined only to within a rigid body motion. As regards the stress field, this is obviously not unique in such regions where the velocity strain vanishes. Where the velocity strain does not vanish, however, we have $s_{ij} = s'_{ij}$. The corresponding stress fields σ_{ij} and σ'_{ij} therefore differ at most in the mean pressure. Since each of these stress fields must satisfy the equations of equilibrium (8), this difference in mean pressure must be constant throughout V. The boundary conditions on S_T, however, rule out such a difference in mean pressure. Thus, the stress field is unique except when $S_T = 0$. In this case the stress field is determined only to within a constant hydrostatic pressure.

5. Minimum principle for velocity strain. A continuous velocity field v_i^* with piecewise continuous first derivatives will be called kinematically admissible if it satisfies the boundary conditions on S_V and the condition of incompressibility (3). Where the corresponding velocity strain ε_{ij}^* does not vanish, Eq. (7) furnishes a field of stress deviation s_{ij}^*. As a rule, however, this field will not satisfy the condition of equilibrium (10).

Excluding the case $S_V = 0$, we propose to compare the unique solution v_i of our boundary value problem to a generic kinematically admissible velocity field v_i^*. Consider the expression

$$H^* = \int (\mu I^{*2} + 2kI^*)dV - 2\int T_i v_i^* dS_T - 2\int F_i v_i^* dV \quad (17)$$

which is a functional of the velocity field v_i^*. Since $v_i^* = v_i$ on S_V, we have

$$\begin{aligned} H^* - H = \int [\mu(I^{*2} - I^2) + 2k(I^* - I)]dV - \\ - 2\int T_i(v_i^* - v_i)dS - 2\int F_i(v_i^* - v_i)dV, \end{aligned} \quad (18)$$

where the surface integral is extended over the entire surface S. We shall prove that $H^* - H > 0$ unless $v_i^* = v_i$, thus establishing the following minimum principle.

Theorem 1. Among all kinematically admissible velocity fields, the actual velocity field minimizes the expression (17).

Using the principle of virtual work and the condition of incompressibility, we write Eq. (18) in the form

$$H^* - H = \int [\mu(I^{*2} - I^2) + 2k(I^* - I) - 2s_{ij}(\varepsilon_{ij}^* - \varepsilon_{ij})]dV, \quad (19)$$

where s_{ij} is the (unique) stress deviation associated with the solution v_i.

In discussing the sign of the integrand B in (19) we must distinguish two types of region.

(i) Where $\varepsilon_{ij} \neq 0$, we have by (7) and (4)

$$B = 2\mu(\varepsilon_{ij}^* - \varepsilon_{ij})(\varepsilon_{ij}^* - \varepsilon_{ij}) + \frac{2k}{I}(II^* - 2\varepsilon_{ij}\varepsilon_{ij}^*). \quad (20)$$

The first term on the right-hand side of (20) is positive unless $\varepsilon_{ij}^* = \varepsilon_{ij}$, and the second term is non-negative by the Schwarzian inequality. Thus, B is positive in region (i) unless $\varepsilon_{ij} = \varepsilon_{ij}^*$.

(ii) When $\varepsilon_{ij} = 0$, the integrand B of (19) reduces to

$$B = \mu I^{*2} + 2kI^* - 2s_{ij}\varepsilon_{ij}^*, \quad (21)$$

where s_{ij} is no longer uniquely determined but must satisfy the condition

$$J \leq k \tag{22}$$

according to (5). If $\varepsilon^*_{ij} = 0$, the expression (21) vanishes. If, on the other hand, $\varepsilon^*_{ij} \neq 0$, the associated stress deviation s^*_{ij} is readily found for (7) and we have

$$J^* > k \tag{23}$$

in accordance with (5). The expression (21) can then be written as follows:

$$B = \frac{J^* - k}{\mu}\left(J^* + k - \frac{s_{ij}s^*_{ij}}{J^*}\right). \tag{24}$$

Now, $s_{ij}s^*_{ij} \leq 2JJ^*$, by the Schwarzian inequality. Equation (24) therefore yields

$$B \geq \frac{J^* - k}{\mu}(J^* + k - 2J) > 0 \tag{25}$$

by (22) and (23).

It follows from the preceding discussion that $H^* > H$ unless $\varepsilon^*_{ij} = \varepsilon_{ij}$ throughout V. In view of the boundary conditions on S_V this means that $H^* > H$ unless $v^*_i = v_i$ throughout V. This establishes the minimum principle of Theorem 1. In the special case of plane flow and under the assumption that the boundary conditions involve only surface tractions and are, moreover, such as to cause visco-plastic deformation throughout V, an equivalent variational principle has been given by Ilyushin [6].

6. Maximum principle for stress. A continuous stress field σ^*_{ij} with piecewise continuous first derivatives will be called statically admissible if it satisfies the boundary condition on S_T and the equations of equilibrium (8). By means of (5) a field of velocity strain ε^*_{ij} is associated with the stress field σ^*_{ij}. As a rule, however, this field of velocity strain does not satisfy the compatibility conditions and hence cannot be derived from a velocity field.

Excluding the case $S_T = 0$, we propose to compare a solution

σ_{ij} of our boundary value problem to a generic statically admissible stress field σ^*_{ij}. It should be kept in mind for the following that two solution stress fields can differ only in the common rigid region.

Consider the expression

$$K^* = 2\int T^*_i v_i dS_V - \frac{1}{4\mu}\int [|J^* - k| + J^* - k]^2 dV \qquad (26)$$

which is a functional of the stress field σ^*_{ij}. Since $T^*_i = T_i$ on S_T, we have

$$K - K^* = 2\int (T_i - T^*_i)v_i dS$$
$$- \frac{1}{4\mu}\int \left\{[|J - k| + J - k]^2 - [|J^* - k| + J^* - k]^2\right\} dV, \quad (27)$$

where the first integral is extended over the entire surface S. We shall prove that $K > K^*$ unless σ^*_{ij} is also a solution stress field in which case $K = K^*$. The following maximum principle will thus be established.

Theorem 2. Among all statically admissible stress fields, the solution stress fields maximize the expression (26).

Using the principle of virtual work and the condition of incompressibility, we write (27) in the form

$$K - K^* = \frac{1}{\mu}\int \left\{2\mu\varepsilon_{ij}(s_{ij} - s^*_{ij}) - \frac{1}{4}[|J - k| + J - k]^2\right.$$
$$\left. + \frac{1}{4}[|J^* - k| + J^* - k]^2\right\} dV, \quad (28)$$

where ε_{ij} is the (unique) velocity strain associated with the solution σ_{ij}.

In discussing the sign of the integrand C in (28) we must distinguish four types of region.

(i) Where $J > k$ and $J^* > k$, we have by (5) and (4)

$$C = \frac{1}{2}(s_{ij} - s^*_{ij})(s_{ij} - s^*_{ij}) - \frac{k}{J}(2JJ^* - s_{ij}s^*_{ij}). \qquad (29)$$

The expression inside the parenthesis in the second term on the right-hand side of (29) is positive unless $s^*_{ij} = cs_{ij}$ in which case it vanishes. In this case, $C > 0$ if $c \neq 1$ and $C = 0$ if $c = 1$. In all other cases C is shown to be positive by the following argument. Since $J > k$ in region (i), we have

$$C > \tfrac{1}{2}(s_{ij}-s^*_{ij})(s_{ij}-s^*_{ij})-(2JJ^*-s_{ij}s^*_{ij})=(J-J^*)^2 \geq 0. \qquad (30)$$

(ii) Where $J > k$ but $J^* \leq k$ we cannot have $s^*_{ij} = s_{ij}$. The integrand C of (28) can be transformed as follows by means of (5) and (4):

$$\begin{aligned} C &= (1 - \tfrac{k}{J})(2J^2 - s_{ij}s^*_{ij}) - (J - k)^2 \\ &= \frac{J - k}{J}(J^2 + kJ - s_{ij}s^*_{ij}). \end{aligned} \qquad (31)$$

The factor in front of the parenthesis on the right-hand side of (31) is positive in region (ii). Moreover, since $J > k \geq J^*$,

$$J^2+kJ-s_{ij}s^*_{ij} > J^2+J^{*2}-s_{ij}s^*_{ij}= \tfrac{1}{2}(s_{ij}-s^*_{ij})(s_{ij}-s^*_{ij}) > 0. \qquad (32)$$

Thus, $C > 0$ in region (ii).

(iii) Where $J \leq k$ but $J^* > k$, we have $\varepsilon_{ij} = 0$. The integrand C in (28) thus assumes the form

$$C = (J^* - k)^2 > 0. \qquad (33)$$

(iv) Where $J \leq k$ and $J^* \leq k$, the integrand in (28) vanishes.

It follows from the preceding discussion that $K > K^*$ unless $s^*_{ij} = s_{ij}$ in all of V but the rigid region (iv). This means that $K > K^*$ unless s^*_{ij} is the stress deviation field associated with a solution. Theorem 2 is thus established.

7. Related extremum principles. The Newtonian viscous fluid and the perfectly plastic Mises solid are limiting cases of the Bingham solid considered here. Indeed, with $k = 0$, Eq. (5) characterizes a Newtonian liquid and, with $\mu = 0$, Eq. (7) characterizes a Mises solid. Accordingly, our extremum principles should contain the extremum principles for the Newtonian liquid and the Mises solid as special cases.

With $k = 0$, the functional (17) reduces to an expression which is readily recognized as the excess of the rate of dissipation of energy in the Newtonian liquid over double the rate at which work is being done by the body forces and the surface tractions on S_T. That this quantity is minimized by the actual velocity field was observed by Helmholtz [7].

With $\mu = 0$, on the other hand, the functional (17) reduces to an expression which is easily identified as double the excess of the rate of dissipation of energy in the Mises solid over the rate at which work is being done by the body forces and the surface tractions on S_T. That this quantity is minimized by the actual velocity field was recognized by Markov [8].

In view of the fact that these special cases of our minimum principle can be stated in terms of energy dissipation, it is worth noting that the first integral on the right-hand side of (17) does not represent the rate of dissipation of energy in the Bingham solid. Indeed, Eq. (7) shows the rate of energy dissipation to be

$$\int s_{ij}\varepsilon_{ij}\, dV = \int (\mu I^2 + kI)\, dV. \qquad (34)$$

As Ilyushin [6] has remarked with reference to the special case considered by him, the first integral on the right-hand side of (17) can be interpreted as the sum of the "power of deformation and the power of the internal forces of plastic resistance". Alternatively, this integral can be considered as the sum of the rate of viscous dissipation of energy and double the rate of plastic dissipation of energy.

With $k = 0$, the functional (26) reduces to an expression which represents the excess of double the rate at which work is being done by the surface tractions on S_V over the rate of dissipation of energy in the Newtonian liquid, the latter rate being expressed in terms of stresses. The principle that this expression is maximized by the actual stress field seems to be new.

Finally, the characteristic equation for a Mises solid in plastic flow is obtained from (5) by letting μ and $J - k$ tend to zero in such a manner that $2\mu J/(J-k)$ tends towards a positive factor of proportionality between s_{ij} and ε_{ij}. If this limiting process is applied to the expression (26), this is seen to reduce to double the rate at which work is being done by the surface tractions on S_V. The fact that this rate of work is maximized by the actual surface tractions was recognized by Hill, first [9] in the special case where the boundary conditions are such as to ensure plastic flow throughout V, and later [10] in the general case.

REFERENCES

1. E. C. Bingham, "Fluidity and Plasticity", McGraw-Hill, New York, 1922, pp. 215-218.

2. R. v. Mises, "Mechanik der festen Koerper im plastisch-deformablen Zustand", Goettinger Nachrichten, math. - phys. Kl. 1913, pp. 582-592 (1913).

3. J. G. Oldroyd, "Two-dimensional plastic flow of a Bingham solid" Proc. Cambridge Phil. Soc. 43, 383-395 (1947); "Rectilinear plastic flow of a Bingham solid", ibid. 43, 396-405; 521-532 (1947), 44, 200-213 (1948).

4. K. Hohenemser and W. Prager, "Ueber die Ansaetze der Mechanik der Kontinua" Z. angew. Math. Mech. 12, 216-226 (1932), p. 224, Eq. (2').

5. W. Prager, "Mécanique des solides isotropes au delà du domaine élastique", Mém. Sci. Math., fasc. 87, Gauthier-Villars, Paris, 1937, p. 27, Eq. (47).

6. A. A. Ilyushin, "The deformation of a visco-plastic solid" (in Russian), Uchenye Zapiski Moskovskogo Gosudarstvennogo Universiteta 39, 1-81 (1940).

7. H. v. Helmholtz, "Zur Theorie der stationaeren Stroeme in reibenden Fluessigkeiten", Verh. naturhist.-med. Vereins Heidelberg 1-7 (1871).

8. A. A. Markov, "On variational principles in the theory of plasticity" (in Russian), Prikladnaia Mat. Mekh. 11, 339-350 (1947).

9. R. Hill, "A variational principle of maximum plastic work in classical plasticity", Q. J. Mech. Appl. Math. 1, 18-28 (1948).

10. R. Hill, "On the state of stress in a plastic-rigid body at the yield point", Phil. Mag. (7) 42, 868-875 (1951).

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