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Columbia University in the City of New York

DEPARTMENT OF CIVIL ENGINEERING

FREE AND FORCED VIBRATIONS OF AN INFINITELY LONG CYLINDRICAL SHELL IN AN INFINITE ACOUSTIC MEDIUM

By

H. H. BLEICH and M. L. BARON

Office of Naval Research

Contract Nonr-266(08)

Technical Report No. 8

CU-8-52 ONR-266(08)-CE

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List of Symbols

θ, z, r	Cylindrical coordinates, Fig. 1
u, v, w	Longitudinal, tangential and radial displacements of the cylindrical shell. Fig. 1. Note that a positive displacement w is inward.
U_k, V_k, W_k	Coefficients defining the shape of modes in vacuo, Eq. (1)
$\phi_{u,k}$ $\phi_{v,k}$ ϕ_w	u, v, and w components of displacement pertaining to the generalized coordinate q_k, Eqs. (2) and (3).
k = 1, 2, 3	Subscript
a	Radius of cylinder
C_k	Coefficient
c	Velocity of sound in medium
dA	Element of surface area of shell
F(r)	r-dependent function in expression for potential
$i = \sqrt{-1}$	
L	Length of longitudinal half-wave of mode of vibration
M_k	Generalized mass (coordinate q_k)
m	Mass of shell per unit area
m_v	Virtual mass of entrained medium
N, N_{uw}, N_{vw}	Coefficients, see Eqs. (40), (46)
n	Number of circumferential waves of mode of vibration
$P = P(t, \theta, z) = \cos n\theta \sin \frac{\pi z}{L} e^{-i\Omega t}$	External radial force (positive if outward)
p	Pressure in medium
p_a	Radial pressure of medium on cylinder
p_o	Static pressure in medium
Q_k	Generalized force (coordinate q_k)
q_k	Generalized coordinate
R	Large value of the radius r
S(t)	Time dependent function in expression for potential
t	Time
w_{st}	w due to static force
$\alpha_k = \frac{W_k^2}{U_k^2 + V_k^2 + W_k^2}$	Coefficient

$\beta, \bar{\beta}$	see Eqs. (19), (37)
Δ	Coefficient, see Eqs. (26), (30), (38) and Tables 1 and 2.
ρ	Mass density of medium
$\Phi = S(t)F(r) \cos n\theta \sin \frac{\pi z}{L}$	Potential function
Ω	Frequency of free or forced vibration of shell surrounded by medium
ω_k	Frequency of cylinder in vacuo

Summary

A method is presented permitting the determination of the frequencies of vibrations of infinitely long thin cylindrical shells in an acoustic medium. (Sections 1 and 3)

Expressions are also obtained for the displacements of the shell and for the pressures p in the medium in the case of forced vibrations due to sinusoidally distributed radial forces. The results indicate that there is a low frequency range where no radiation takes place, and a high frequency range where the external force provides energy which is radiated. Resonance occurs in the low frequencies range only, in the high range it is prevented by the damping due to radiation.

Free and forced vibrations of steel shells submerged in water are discussed in detail (Sections 3 and 4). With limitations, the theory may be applied approximately to stiffened shells.

The method requires only a minor modification to account for the effect of static pressure in the surrounding medium (Section 5).

The results are also used to draw a general conclusion concerning the treatment of transient problems, (Section 6), particularly of the problem of shock loads.

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I. Free Vibrations

The free and forced vibrations of a submerged infinitely long, thin cylindrical shell are studied in this paper by considering the shell without the fluid as a separate structure responding to the applied forces combined with the dynamic forces exerted by the surrounding infinite acoustic medium. Using the modes of free vibration of the shell in vacuo as generalized coordinates, its response can be expressed in terms of the infinite number of these modes.

The frequencies and modes of vibrations of infinitely long thin cylindrical shells required for the purpose were determined in ref. 1. The individual modes can be classified by the length L of the longitudinal half-wave of the displacements and by the integral number n of circumferential waves. For each length L and number n there exist three frequencies $\omega_k (k = 1, 2, 3)$. The displacements of the modes corresponding to all three frequencies are sinusoidal, but the ratios of longitudinal, circumferential and radial displacements, u, v and w differ. Excluding the case n = 0, the displacements corresponding to the frequency ω_k may be written:

$$
\begin{aligned}
u &= U_k \cos n\theta \cos \frac{\pi z}{L} \\
v &= V_k \sin n\theta \sin \frac{\pi z}{L} \qquad (1) \\
w &= W_k \cos n\theta \sin \frac{\pi z}{L}
\end{aligned}
$$

The modes defined in Eqs. (1) contain an arbitrary factor, only the ratios V_k/U_k and W_k/U_k can be determined and are contained in ref. 1. It might also be mentioned that Eqs. (1) can be generalized by adding phase angles, which is equivalent to changing the origin of the coordinates θ and z.

The most general response of a cylindrical shell will be a combination of the modes for all values of L and n. However, in order to find for a submerged shell the frequencies and modes of free vibration of selected values L and n, one need consider only the three modes of shape (1) having the same parameters L and n. This simplification is due to the fact that the wave equation (8) for the surrounding medium has a solution containing the factor $\cos n\theta \sin \frac{\pi z}{L}$, and the pressure belonging to such a solution does not excite modes of different values L or n.

Selecting the coefficients W_k in Eqs. (1) as generalized coordinates $W_k = q_k$, and using the abbreviations

$$
\begin{aligned}
\phi_{u,k} &= \frac{U_k}{W_k} \cos n\theta \cos \frac{\pi z}{L} \\
\phi_{v,k} &= \frac{V_k}{W_k} \sin n\theta \sin \frac{\pi z}{L} \qquad (2) \\
\phi_w &= \cos n\theta \sin \frac{\pi z}{L}
\end{aligned}
$$

the displacements u, v and w can be expressed as functions of the three generalized coordinates q_1, q_2, q_3

$$u = q_1\phi_{u,1} + q_2\phi_{u,2} + q_3\phi_{u,3} \tag{3a}$$

$$v = q_1\phi_{v,1} + q_2\phi_{v,2} + q_3\phi_{v,3} \tag{3b}$$

$$w = (q_1 + q_2 + q_3)\phi_w \tag{3c}$$

The symbol ϕ_w in Eqs. (2) and (3) does not have a second subscript, as there is no difference in the w - displacements of the three coordinates q_k.

Due to the inherent orthogonality between the displacements represented by the generalized coordinates, the equations of motion contain each only one of the coordinates q_k,

$$M_k\ddot{q}_k + M_k\omega_k^2 q_k = Q_k \qquad (k = 1, 2, 3) \tag{4}$$

where the generalized mass M_k is

$$M_k = \iint_A m(\phi_{u,k}^2 + \phi_{v,k}^2 + \phi_w^2)dA \tag{5}$$

In the case of free vibrations the generalized force Q_k is due solely to the radial pressure p_a of the medium on the cylinder at r = a:

$$Q_k = \iint_A p_a\phi_w dA \tag{6}$$

Hence, Eq. (4) becomes

$$\ddot{q}_k + \omega_k^2 q_k = \frac{\iint_A p_a\phi_w dA}{m\iint_A (\phi_{u,k}^2 + \phi_{v,k}^2 + \phi_w^2)dA} \tag{7}$$

The integrals in Eq. (5) to (7) are to be taken over the surface of a section of the cylindrical shell of length L.

The shell is assumed to be submerged in an acoustic medium, the velocity potential Φ for which is governed by the equation

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \tag{8}$$

A solution of this equation suitable for the present problem is

$$\Phi = S(t)F(r)\cos n\theta \sin\frac{\pi z}{L} = S(t)F(r)\phi_w \tag{9}$$

where S(t) and F(r) are functions at our disposal. At the surface of the shell (r = a), the radial velocity of the shell must equal that of the medium:

$$\dot{w} = -\frac{\partial\Phi}{\partial r}\Big]_{r=a} \tag{10}$$

Substituting Eqs. (3c) and (9) one obtains

$$(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)\phi_w = -S(t)F'(a)\phi_w$$

or

$$S(t) = \frac{-(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)}{F'(a)} \tag{11}$$

(Note: $F'(a) = \frac{dF(r)}{dr}\Big]_{r=a}$).

The radial pressure exerted on the shell by the surrounding medium is the pressure at $r = a$:

$$p_a = -\rho\frac{\partial\Phi}{\partial t}\Big]_{r=a} = -\rho\dot{S}(t)F(a)\phi_w \tag{12}$$

or substituting Eq. (11),

$$p_a = \rho\frac{F(a)}{F'(a)}\phi_w(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3) \tag{13}$$

Eqs. (6) and (7) become

$$Q_k = \rho\frac{F(a)}{F'(a)}(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3)\iint_A \phi_w^2 dA \tag{14}$$

and

$$\ddot{q}_k + \omega_k^2 q_k = \frac{\rho\alpha_k}{m}\frac{F(a)}{F'(a)}(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3) \tag{15}$$

where

$$\alpha_k = \frac{\iint_A \phi_w^2 dA}{\iint_A (\phi_{u,k}^2 + \phi_{v,k}^2 + \phi_w^2) dA} = \frac{W_k^2}{U_k^2 + V_k^2 + W_k^2} \tag{16}$$

The last equation is obtained by substituting Eq. (2). Numerical values of α_k for various modes are given in the tables of ref. 1.

The function F(r) is obtained from Eq. (8), which reads in full

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \tag{8a}$$

Assuming periodic solutions for the generalized coordinates q_k

$$q_k = C_k e^{i\Omega t} \tag{17}$$

Eqs. (9) and 11) give

$$\Phi = -i\Omega \frac{F(r)}{F'(a)} \phi_w (C_1 + C_2 + C_3) e^{i\Omega t} \tag{18}$$

Introducing the symbol

$$\beta = \sqrt{\frac{\pi^2}{L^2} - \frac{\Omega^2}{c^2}} \tag{19}$$

Eqs. (8a) and (18) lead to the differential equation

$$F''(r) + \frac{1}{r} F'(r) - \left(\beta^2 + \frac{n^2}{r^2}\right) F(r) = 0 \tag{20}$$

As boundary conditions we require that the velocity of the medium at $r = \infty$ vanishes in such a manner that the kinetic energy in the medium remains finite,

$$r^{\frac{3}{2}} F(r) \Big]_{r=\infty} = 0 \tag{21}$$

A solution of Eq. (20) is

$$F(r) = H_n^{(1)} (i\beta r) \tag{22}$$

which satisfies Eq. (21) provided the real part of β is positive.
Computing

$$F'(r) = i\beta \left[\frac{-n}{i\beta r} H_n^{(1)} (i\beta r) + H_{n-1}^{(1)} (i\beta r) \right] \tag{23}$$

and

$$\frac{F(a)}{F'(a)} = \frac{-a}{n - i\beta a \dfrac{H_{n-1}^{(1)} (i\beta a)}{H_n^{(1)} (i\beta a)}} \tag{24}$$

Eq. (15) becomes:

$$\ddot{q}_k + \omega_k^2 q_k = -\frac{\rho a}{2m} \alpha_k \Delta (\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3) \tag{25}$$

where

$$\Delta = \Delta(\beta a) = \frac{-2}{a} \frac{F(a)}{F'(a)} = \frac{2}{n - i\beta a \dfrac{H_{n-1}^{(1)} (i\beta a)}{H_n^{(1)} (i\beta a)}} \tag{26}$$

Substituting Eqs. (17) into (25) three homogeneous equations in the coefficients C_k are obtained

$$(\omega_k^2 - \Omega^2) C_k - \alpha_k \frac{\rho a}{2m} \Omega^2 \Delta (C_1 + C_2 + C_3) = 0 \qquad (k = 1, 2, 3) \tag{27}$$

Non-vanishing solutions of these equations exist only if their determinant vanishes, which after evaluation leads to the frequency equation

$$\frac{a_1}{\omega_1^2 - \Omega^2} + \frac{a_2}{\omega_2^2 - \Omega^2} + \frac{a_3}{\omega_3^2 - \Omega^2} = \frac{2m}{\rho a \Omega^2 \Delta} \tag{28}$$

valid for $n \neq 0$.

For the previously excluded case, $n = 0$, the cylindrical shell in vacuo has again three modes for each value L and n. One of these modes is, however, a purely torsional motion with displacement components $u = w = 0$. This mode is not excited by the radial forces from the surrounding medium, and only the two non-torsional modes need be considered. Instead of Eq. (25) one obtains for $n = 0$:

$$\ddot{q}_k + \omega_k^2 q_k = -\frac{\rho a}{2m} a_k \Delta (\ddot{q}_1 + \ddot{q}_2) \tag{29}$$

where

$$\Delta = -2 \frac{i H_0^{(1)}(i\beta a)}{\beta a \, H_1^{(1)}(i\beta a)} \tag{30}$$

The frequency equation for $n = 0$ is:

$$\frac{a_1}{\omega_1^2 - \Omega^2} + \frac{a_2}{\omega_2^2 - \Omega^2} = \frac{2m}{\rho a \Omega^2 \Delta} \tag{31}$$

The solution of the frequency equations (28) and (31) is further discussed in Section 3.

2. Forced Vibrations due to Radial Forces

Any arbitrarily distributed radial force $P(t, \theta, z)$ can be expanded in a Fourier integral (or series) of terms

$$P(t, \theta, z) = e^{i\Omega t} \cos n\theta \sin \frac{\pi z}{L} = e^{i\Omega t} \phi_w \tag{32}$$

The following treatment will be restricted to the consideration of such individual terms, i.e. to force components $P(t, \theta, z)$ defined by Eq. (32). These forces P are counted positive if acting outwards. Non-radial forces can be treated in a similar manner but will not be considered.

Eqs. (4) and (5) of Section 1 apply again, but Eq. (6) for the generalized force Q_k must be replaced by

$$Q_k = \int p_a \phi_w dA - \int e^{i\Omega t} \phi_w^2 dA \tag{33}$$

Proceeding essentially in the same manner as in Section 1, using Eqs. (8) to (13) incl., and Eq. (16) to (24) incl., the equations of motion for $n \neq 0$ become

$$\ddot{q}_k + \omega_k^2 q_k = -\frac{\rho a}{2m}\alpha_k \Delta(\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3) - \frac{\alpha_k}{m} e^{i\Omega t} \qquad (k = 1, 2, 3) \tag{34}$$

There is just one detail in the derivation of this equation which requires elucidation. The solution (22)

$$F(r) = H_n^{(1)}(i\beta r)$$

does not satisfy the boundary condition (21) for frequencies $\Omega > \frac{\pi c}{L}$ because the argument

$$a\beta = a\sqrt{\frac{\pi^2}{L^2} - \frac{\Omega^2}{c^2}}$$

becomes purely imaginary. The boundary condition (21) is in this case replaced by the requirement that the potential function Eq. (18)

$$\Phi = -\frac{i\Omega}{F'(a)}(C_1 + C_2 + C_3)e^{i\Omega t} F(r)\phi_w \tag{35}$$

represents an outgoing wave for large values of r. This condition is satisfied for

$$F(r) = H_n^{(2)}(\bar{\beta} r) \tag{36}$$

where

$$\bar{\beta} = \sqrt{\frac{\Omega^2}{c^2} - \frac{\pi^2}{L^2}} \qquad \left(\Omega \geq \frac{\pi c}{L}\right) \tag{37}$$

and the positive sign of the square root applies. This leads to the value

$$\Delta = \Delta(\bar{\beta} a) = -\frac{2}{a}\frac{F(a)}{F'(a)} = \frac{2}{n - \bar{\beta} a \dfrac{H_{n-1}^{(2)}(\bar{\beta} a)}{H_n^{(2)}(\bar{\beta} a)}} \tag{38}$$

valid for $\Omega \geq \frac{\pi c}{L}$, while for $\Omega \leq \frac{\pi c}{L}$ Eq. (26) remains in force.

Substitution of Eq. (17)

$$q_k = C_k e^{i\Omega t}$$

in the equations of motion (34), leads to three non-homogeneous linear equations for the coefficient C_k. Their solution is

$$C_k = \frac{\dfrac{\alpha_k}{\omega_k^2 - \Omega^2}}{m\left(\dfrac{\rho a}{2m} N\Delta - 1\right)} \tag{39}$$

where

$$N = \frac{a_1 \Omega^2}{\omega_1^2 - \Omega^2} + \frac{a_2 \Omega^2}{\omega_2^2 - \Omega^2} + \frac{a_3 \Omega^2}{\omega_3^2 - \Omega^2} \tag{40}$$

Substituting the values C_k from Eq. (39) into the expression (18) for the potential function Φ and noting Eqs. (26), (32) and (38) leads to

$$\Phi = \frac{i \Omega a \Delta}{2m} \left[\frac{1}{\left[\frac{\rho a}{2m} \Omega^2 \Delta - \frac{\Omega^2}{N} \right]} \right] \frac{F(r)}{F(a)} P(t, \theta, z) \tag{41}$$

where the function $F(r)$ is given by Eqs. (22) and (36), respectively,

$$F(r) = \begin{cases} H_n^{(1)}(i\beta r) & \text{if } \Omega < \frac{\pi c}{L} \quad (42a) \\ r^{-n} & \text{if } \Omega = \frac{\pi c}{L} \text{ and } n \neq 0 \quad (42b) \\ H_n^{(2)}(\bar{\beta} r) & \text{if } \Omega > \frac{\pi c}{L} \quad (42c) \end{cases}$$

Δ, β and $\bar{\beta}$ are defined by Eqs. (19), (26), (37) and (38).

The pressure in the medium, $p = - \rho \frac{\partial \Phi}{\partial t}$, becomes

$$p = \frac{1}{1 - \frac{2m}{\rho a N \Delta}} \frac{F(r)}{F(a)} P(t, \theta, z) \tag{43}$$

The pressure p_a on the surface of the cylinder is

$$p_a = \frac{P(t, \theta, z)}{1 - \frac{2m}{\rho a N \Delta}} \tag{43a}$$

The knowledge of the coefficients C_k permits also the determination of the displacements of the cylinder. Of particular interest are the radial displacements w obtained from Eq. (3c)

$$w = \frac{1}{m \Omega^2} \frac{P(t, \theta, z)}{\left[\frac{\rho a}{2m} \Delta - \frac{1}{N} \right]} \tag{44}$$

If one goes to the limit $\Omega \to 0$ this equation gives an expression for the deflection w_{st} under static loading $P(\theta, z) = \phi_w$

$$w_{st} = \lim_{\Omega \to 0} w = - \left[\frac{a_1}{\omega_1^2} + \frac{a_2}{\omega_2^2} + \frac{a_3}{\omega_3^2} \right] \frac{P}{m} \tag{44a}$$

while a similar process for the pressure leads to the obvious result $\lim_{\Omega \to 0} p = 0$.

The expressions for the longitudinal and tangential displacements u and v due to the radial force $P(t, \theta, z)$ are somewhat more complicated

$$u = \frac{N_{uw}}{mN\Omega^2} \, \frac{\cos n\theta \cos \frac{\pi z}{L} e^{i\Omega t}}{\frac{\rho a}{2m}\Delta - \frac{1}{N}}$$

$$v = \frac{N_{vw}}{mN\Omega^2} \, \frac{\sin n\theta \sin \frac{\pi z}{L} e^{i\Omega t}}{\frac{\rho a}{2m}\Delta - \frac{1}{N}} \tag{45}$$

where

$$N_{uw} = \sum_{k=1}^{3} \frac{a_k \frac{U_k}{W_k}\Omega^2}{\omega_k^2 - \Omega^2}$$

$$N_{vw} = \sum_{k=1}^{3} \frac{a_k \frac{V_k}{W_k}\Omega^2}{\omega_k^2 - \Omega^2} \tag{46}$$

The above Eqs. (40) to (46) were derived for the case $n \neq 0$, but with exception of Eq. (42b), they remain valid for $n = 0$ if the terms referring to the non-existing frequency ω_3 are omitted when computing N, N_{uw}, and N_{vw} from Eqs. (40) to (46).

The degenerate solutions (42b) for $n = 0$ and for $n = 1$ do not satisfy Eq. (21) and the total kinetic energy in the medium is in these cases not finite. This indicates that steady state vibrations for $n = 0$ or 1 in an actually infinite medium are not possible if $\Omega = \frac{\pi c}{L}$. If one considers the infinite boundary of the medium as an approximation of a far boundary, $r = R$, the solution (42b) for $n = 1$ may be considered an approximation of physical meaning because the energy is then finite and the velocities at the boundary are small, as $1/R$. A similar reasoning does not apply for $n = 0$, because the longitudinal velocity $\frac{\partial \Phi}{\partial z}$ is not small for large values of r; the physical impossibility of this solution expresses itself in the fact that $\Delta = \infty$. To obtain a physical meaningful solution for $n = 0$ it is necessary to consider the actual conditions at the distant boundary.

To facilitate numerical computations, Tables 1 and 2 list the values of Δ occuring in Eqs. (41) to (46) for $n = 0$ to 4 incl.*

3. Discussion of the Frequency Equation

The discussion can be restricted to the frequency equation

*Values of Δ for $n \leq 20$ can be obtained from ref. 4, but only if $\Omega > \frac{\pi c}{L}$. The acoustic reactance and resistance ratios χ_n and θ_n shown in Fig. 1 and 2 of ref. 4 can be used to compute $a\bar{\beta}\Delta = \chi_n(a\bar{\beta}) - i\theta_n(a\bar{\beta})$.

$\frac{1}{2} a\bar{\beta}\Delta$

$$\frac{a_1}{\omega_1^2 - \Omega^2} + \frac{a_2}{\omega_2^2 - \Omega^2} + \frac{a_3}{\omega_3^2 - \Omega^2} = \frac{2m}{\rho a \Omega^2 \Delta} \tag{28}$$

as the case [illegible] 0, Eq. ([illegible]) may be obtained by substituting $a_3 = 0$. This equation is transcende[illegible] th[illegible] parameter [illegible] being a transcendental function of the frequency Ω.

<u>Number [illegible] Ro[illegible]ots of Frequency Equation.</u>

The paramet[illegible]r Δ is [illegible] function of

$$a\beta = a\sqrt{\frac{\pi^2}{L^2} - \frac{\Omega^2}{c^2}} \tag{47}$$

and examination sh[illegible]ows that the values of Δ are real if $a\beta$ is real, but that Δ becomes complex if $\Omega > \frac{\pi c}{L}$, so that $a\beta$ is imaginary. This leads to the question how many real roots Ω Eq. (28) wil[illegible] furnish. The coefficients a_k and the frequencies ω_k on the left hand side of Eq. (28) being [illegible]eal, the entire left hand side will always be real. As $\frac{2m}{\rho a}$ is also real Eq. (28) can have [illegible] solution for Ω when $a\beta$ is imaginary, that is any real root Ω must be below the lim[illegible] $\Omega \leq \frac{\pi c}{L}$.

Further information [illegible]n the number of real roots can be obtained by considering Eq. (28) graphically. For n [illegible] 0, Fig. 2 shows in solid lines a typical plot of the values of the left hand side of Eq. (2[illegible]) as function of Ω. Starting with a positive value for $\Omega = 0$, the plot shows vertical a[illegible]symptotes for each of the values $\Omega = \omega_1, \omega_2$ and ω_3. The right hand side of Eq. (28) is [illegible]ted in dashed lines, the intersections indicating roots of the frequency equation. As [illegible] right hand side has real values only for $0 \leq \Omega \leq \frac{\pi c}{L}$ the curve ends at point E. The [illegible]lot indicates that there are at least as many real roots Ω as there are frequencies ω_k of the system in vacuo which lie below $\frac{\pi c}{L}$, and at the most one more. This latter cas[illegible] can occur if there is a further frequency ω_k just larger than $\frac{\pi c}{L}$, so that the endpoint [illegible] is in a position like E' in Fig. 2. If in particular all frequencies ω_k are larger than [illegible] $\frac{\pi c}{L}$, there may be no real frequency at all. The case of no real root will occur only for mediums having very low density [illegible].

For n = 0 somewhat similar conditions prevail, Fig. 3. In this case the end point E lies however on the Ω - ax[illegible]is, which results in the occurence of one real root in all cases, even if $\omega_1 > \frac{\pi c}{L}$. An example of such a case is a steel shell in water vibrating with a half wave length L > [illegible]

It is also apparent fr[illegible]om Figs. 2 and 3 that the lowest root Ω_1 of the frequency equation must be smaller than the frequency ω_1, while the next root Ω_2, if any, must lie between ω_1 and ω_2 and th[illegible]e highest root must be between ω_2 and ω_3. No root can lie above the highest freque[illegible]ncy ω_3 in vacuo.

Attention is drawn [illegible] the fact that due to the orthogonality of the modes q_k of the shell in vacuo

$$a_1 + a_2 + a_3 = 1 \tag{48}$$

As a rule one, and for many configurations two values a_k are quite small, indicating that the radial displacements of such modes are small. In such cases the coupling of the motion of the shell and the surrounding medium is slight, and the frequencies Ω of such modes will be very close to the frequencies ω in vacuo.

To simplify the numerical determination of the roots of the frequency equation Table 1, giving values of Δ for real arguments $a\beta$, has been provided. After a frequency Ω has been obtained the shape of the mode can be determined by computing the coefficient C_k from Eqs. (27).

Effect of Compressibility

It is of considerable interest to consider the effect of the compressibility in order to determine when it might be neglected. The compressibility appears in the derivation of Eq. (8) where the velocity of sound c occurs, and the solution for an incompressible fluid may be obtained by using $c = \infty$. This does not change the frequency equation (28), but the argument $a\beta$ used in determining the term Δ in this equation is in this case

$$a\beta = \frac{a\pi}{L} \tag{49}$$

instead of the value given in Eq. (47) which depended on the frequency Ω. Comparing Eqs. (47) and (49), it is apparent that the compressibility may be neglected only if $\frac{\Omega}{c} << \frac{\pi}{L}$.

If the compressibility is neglected the number of real roots Ω is always equal to the number of frequencies ω_k. It is obvious that roots higher than $\frac{\pi c}{L}$ found for the incompressible case are unreal.

Complex Roots of Frequency Equation

The fact that the number of real roots Ω of the frequency equation may be smaller than the number of frequencies ω_k in vacuo raises the question of the existence of complex roots in lieu of the missing real roots, and of the physical meaning of complex roots if any. In order to have physical meaning the solution corresponding to a complex root Ω must represent an outgoing wave, and must also decay with time.

In order to recognize the wave character of the solution it is convenient to use the potential function in the form (36) employed in Section 2 for $\Omega > \frac{\pi c}{L}$,

$$\Phi = Ce^{i\Omega t} H_n^{(2)}(\bar{\beta} r)\, \phi_w \tag{50}$$

To avoid incoming waves the real part of $\bar{\beta}$ must not be negative. Based on this expression one obtains a "wave frequency equation", which is identical with Eqs. (28) or (31), except that the value Δ is a function of $a\bar{\beta}$ and must be computed from Eq. (38) instead of Eq. (26). It is to be noted that the argument $a\bar{\beta}$ in Eq. (38) will now be complex. A study of the potential function (50) indicates that for complex roots Ω for which the time dependent term $e^{i\Omega t}$ is damped, the term $H_n^{(2)}(\bar{\beta} r)$ becomes ∞ at $r \to \infty$. Such roots and the corresponding solutions appear at first to have no physical significance. It should be pointed out however, that these complex roots and the corresponding solutions occur and have significance as <u>asymptotic solutions</u> in transient problems. In such cases the validity of the asymptotic solution is limited to finite values of r and the fact that $H_n^{(2)}(\overline{\beta}r)$ does become infinite at $r \to \infty$ is immaterial. For example, a heavy shell in a light medium, like air, is capable of slightly damped vibrations. The frequency equation in such a case has complex roots Ω the real parts of which are close to the frequencies ω_k in vacuo, while the imaginary parts are quite small. Expression (50) is then the approximation of a possible motion, valid in the range $t > T$ and $r < R$, where T must be selected so large that the transient effects of the initiation of the motion have passed outside the cylinder having the arbitrarily selected radius R.

The numerical values of complex frequencies Ω can easily be determined approximately if, either the imaginary part of Ω is small, or if $|\Omega|$ is large. If the imaginary part of Ω is small the value of $\Delta(a\overline{\beta})$ computed for the adjoining real part of Ω, found in Table 2, may be used. This procedure would be applicable for the above mentioned case of a heavy shell in a light medium. If the absolute value of the frequency $|\Omega|$ is large such that the argument $|a\overline{\beta}| >> 1$, the term $\Delta(a\overline{\beta})$ may be determined by using asymptotic expressions for the Hankel Functions,

$$\Delta(a\overline{\beta}) = \frac{4}{1 + 2ia\overline{\beta}} \tag{51}$$

Frequently it is permissible to neglect even the real part of the denominator in this equation, giving

$$\Delta(a\overline{\beta}) = -\frac{2i}{a\overline{\beta}} \tag{52}$$

These approximations apply for all values of n, but give better results for small values of n. A measure of the degree of approximation can be obtained by comparing the results of Eqs. (51) (52), with the values for real arguments $a\overline{\beta}$ in Table 2.

<u>Virtual Mass</u>

For [illegible] it is possible to simplify the frequency equation by neglecting the two higher modes, thus obtaining an approximate equation for the lowest frequency. This equation can then be interpreted in a simple manner using the concept of the " virtual mass" of the entrained medium.

The numerical values of a_k and ω_k occuring in Eq. (28) are given in ref. 1. These values indicate [illegible] within the range $1 \leq \frac{L}{a} \leq 10$ the first term of Eq. (28), $a_1/(\omega_1^2 - \Omega^2)$, is very much larger than the other two if $\Omega < \omega_1$; for $\frac{L}{a}$ values near 1 this is so because a_2 and a_3 are small, for large $\frac{L}{a}$ values because ω_2^2 and ω_3^2 are very much larger than ω_1^2. Neglecting the terms with a_2 and a_3 leads after rearrangement to

$$\Omega^2 \left(1 + \frac{a_1 \rho a}{2m} \Delta\right) = \omega_1^2 \tag{53}$$

This indicates that the lowest frequency Ω of the submerged shell is equal to the frequency of a shell in vacuo whose mass has been increased by the " virtual mass" m_v of a portion of the medium vibrating with it

$$m_v = \alpha_1 \frac{\rho a}{2} \Delta \tag{54}$$

Δ is a function of the frequency given by Eqs. (19) and (26), but if $\omega_1 << \frac{\pi c}{L}$ the effect of the frequency is negligible and Eq. (49) for the incompressible case may be used instead of Eq. (19). m_v may be computed quickly by the use of Table 1.

The same approach can be used also for n = 0, but gives good results only if $\frac{L}{a} < 2$. For longer shells the second term in Eq. (31) remains large and can not be neglected, thus making the simple virtual mass concept inapplicable.

Steel Shells Submerged in Water

When applying the frequency equations to thin steel shells (say $\frac{a}{t} > 30$) submerged in an infinite body of water it is found that for ratios $1 \leqslant \frac{L}{a} \leqslant 10$ and $n > 1$ one and only one frequency in vacuo is lower than $\frac{\pi c}{L}$, while for $n = 0$ all frequencies ω_k are above this limit.* It follows that for $n = 0$ one and only one real root Ω exists; for $n > 0$ there is at least one root and according to the previous discussion the possibility of a second one. A second root Ω can only occur, see E' in Fig. 3, if the second frequency ω_2 is close to $\frac{\pi c}{L}$. Ref. 1 shows that in the range $1 \leqslant \frac{L}{a} \leqslant 10$ the second frequency ω_2 is much larger than the limit $\frac{\pi c}{L}$, and there will therefore be just one real frequency for each value of n.** This frequency can be determined from the simplified Eq. (53), except if $n = 0$ and $\frac{L}{a} > 2$.

Similar conditions prevail for ratios $L/a > 10$, except for $n \geqslant 2$, where for extremely large ratios, due to the effect of the bending stiffness of the shell all the frequencies in vacuo may lie above $\frac{\pi c}{L}$, resulting in no real roots. In the limiting case $L = \infty$ only complex roots occur for any value of n. This case has been discussed independently in ref. 2.

If $\frac{L}{a} < 1$ the above result is expected to remain valid, unless the length of the half wave is so short, comparable to the thickness t, that all frequencies in vacuo lie above $\frac{\pi c}{L}$.

Complex roots of the frequency equation representing waves exist and can be determined approximately by using the asymptotic values for $\Delta(a\bar{\beta})$, Eq. (52). However the solutions corresponding to the complex roots lack physical meaning, except as asymptotic solutions of transient problems discussed in the previous section. In the

*The frequencies in vacuo for $1 \leqslant \frac{L}{a} \leqslant 10$ can be obtained from ref. 1.

**This count excludes the frequency of the purely torsional mode of the shell for n = 0. This mode was excluded when deriving Eq. (31) as it is not coupled with the surrounding medium.

case of a steel shell in water where $1 \leq \frac{L}{a} \leq 10$, the asymptotic solution is a combination of the free vibration, corresponding to the real frequency Ω, and of damped wave motions corresponding to the complex roots Ω. If the motion is such that the undamped motion is at all excited, the asymptotic solution converges rapidly toward the undamped vibration of real frequency Ω and the damped motions due to complex frequencies will give only unimportant contributions.

4. Discussion of Results for Forced Vibrations

Response of Shell and Medium

The most significant feature of the results of Section 2, is the fact that the response is entirely different for frequencies Ω of the applied force above and below the limit $\Omega = \frac{\pi c}{L}$. Different expressions, see Eqs. (42 a - c), apply in the two ranges, resulting in different physical behavior.

If $\Omega \leq \frac{\pi c}{L}$ the ratios $H_n^{(1)}(i\beta r)/H_n^{(1)}(i\beta a)$, and as a consequence the values of the coefficient $\Delta(a\beta)$, are real. Eqs. (43) to (45) indicate that in this case the pressure p and the displacements are in phase with the applied force P. There is no input of energy by the applied force and no radiation. It is of interest that the pressure p decreases very rapidly with increasing radius because $H_n^{(1)}(i\beta r)$ decreases like $e^{-\beta r}/\sqrt{r}$. As expected the expressions for the pressure and displacements contain a denominator which vanishes if Ω becomes equal to one of the roots of the frequency equation for free vibrations, indicating resonance.

If the frequency Ω is sufficiently smaller than $\frac{\pi c}{L}$ the effect of the compressibility becomes unimportant and the approximate Eq. (49) for $a\beta$ may be used. For low frequencies, $\Omega \lesssim \omega_1$ one can neglect the modes q_2 and q_3. The resulting equation can be interpreted by means of the virtual mass defined by Eq. (54), indicating that the response is identical with that of a shell of increased mass $(m + m_v)$ in vacuo.

If $\Omega > \frac{\pi c}{L}$ conditions are quite different. The ratio $H_n^{(2)}(\bar{\beta}r)/H_n^{(2)}(\bar{\beta}a)$ is complex and the coefficient $\Delta(a\bar{\beta})$ also is complex. As a result, the pressure p and the displacements are not in phase with the applied force, the phase angle being such that the applied force supplies energy which is radiated. There are no real roots of the frequency equation above $\frac{\pi c}{L}$, and resonance does not occur in this range.

The range $\Omega > \frac{\pi c}{L}$ can be further subdivided in two distinct ranges if $n \neq 0$. Table 2 indicates that for small values of $a\bar{\beta}$ the imaginary part of Δ is very much smaller than the real one, while for large values of $a\bar{\beta}$ the opposite is true. The dividing line is quite sharp, but depends on the value of n. For $n \geq 2$ it lies above $a\bar{\beta} = n/2$. This indicates that, particularly if $n \geq 2$, there is an extensive range of frequencies $\Omega > \frac{\pi c}{L}$ where radiation and damping are quite small. The decay of the pressure p as function of r is also different in the two ranges. If the radiation is large the pressure decays very slowly, approximately as $1/\sqrt{r}$. If the radiation is small, the pressure decays much faster, nearly as fast as in the non-radiating case (although for very large values of r the decay becomes again $1/\sqrt{r}$).

Considering the response as function of the forcing frequency Ω, a number of significant frequencies, in addition to frequencies of free vibrations, can be found by considering the denominators of Eqs. (43) to (45). If the frequency Ω is equal to one of the frequencies ω_k in vacuo, $N = \infty$, and Eq. (44) gives $p_0 = P(t, \theta, z)$; the pressure in the medium is in this case equal to the applied pressure, as if the shell were non-existent. This is physically understandable, because at such a frequency the shell vibrates without requiring any outside force, and the entire applied force is transmitted to the medium.

Other significant values of the frequency Ω are those for which the term N vanishes. Consideration of Eqs. (40) and (46) shows that this occurs, in addition to the trivial value $\Omega = 0$, once between successive values of the frequencies ω_k in vacuo. If $N = 0$, both p and w vanish everywhere, indicating that there is no response at all of the fluid at such frequencies. This rather startling result is due to the fact that for these frequencies the three modes (or two modes if $n = 0$) interact in the manner of a vibration damper. If the longitudinal and radial displacements u and v are determined it is seen that they do not vanish, only w and p are zero. The frequencies at which this phenomenon occurs are necessarily the same as for the shell in vacuo.

At the end of Section 2 it has been pointed out that, and for what reason, the solution for $n = 0$ loses its meaning for $\Omega = \frac{\pi c}{L}$. The formulas for the response for frequencies very close to $\Omega = \frac{\pi c}{L}$ must also be used with caution, because they might be affected by the conditions at the distant boundaries of the medium.

Steel Shells in Water

Considering cases in the range $1 \leq \frac{L}{a} \leq 10$, there is for $n \geq 1$ one frequency in vacuo, ω_1, below $\frac{\pi c}{L}$, while the other two, ω_2, ω_3, are much higher. The frequency equation of the submerged shell has one real root, $\Omega_1 < \omega_1$, and two complex ones near ω_2 and ω_3.

Figs. 4 and 5 show the amplitudes of the radial response w and the maximum pressure p_a at the surface of the shell in non-dimensional manner for a typical case, $n = 2$, $\frac{L}{a} = 4$, $\frac{\rho a}{m} = 4$. (The ratio $\frac{\rho a}{m} = 4$ indicates that the mass of the water displaced by the shell is 2.0 times the mass of the shell) Fig. 4 shows in solid lines the ratio* $|w/w_{static}|$ of the amplitude of the displacement w to the static deflection of the shell as function of the ratio $\frac{\Omega L}{\pi c}$. Attention is drawn to the fact that the right half of the diagram is enlarged and shows $100|w/w_{static}|$. The same figure shows for comparison in dashed lines the response of the shell in vacuo. Fig. 5 shows the ratio of the peak values of the surface pressure p_a to the peak value P of the applied pressure $P(t, \theta, z)$.

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*Often called amplification.

The diagrams show clearly the previously mentioned fact that w and p become zero at two points, where N = 0, which is due to the vibration damper effect. The difference between the response in vacuo and in water for $\frac{\Omega L}{\pi c} > 1$ as shown in Fig. 4 is essentially that between an undamped and damped system, the response curve showing peaks near the undamped natural frequencies ω_k .

The principal point of interest in the response curve for the pressure, $\left|\frac{p_a}{P}\right|$, Fig. 5, is its sawtooth pattern. There is the expected resonance at the one real root Ω_1, followed by a minor minimum near $\frac{\Omega L}{\pi c} = 1$; then, near ω_2 and ω_3, there are zero values followed closely by peaks of unit value. It is also of interest that the pressure decreases with increasing frequencies only slowly, much slower than the response w.

Figs. 6 and 7 show the radial response w and the pressure p_a for the case n = 0. The difference between the responses w in vacuo and in water is much more pronounced as in the case n = 2. This is principally due to the fact that the only frequency Ω_1 of the submerged shell is very much lower than the fundamental frequency ω_1 of the shell in vacuo. Attention is drawn to the extreme narrowness of the peaks of the curves in Figs. 6 and 7 at the resonant frequency Ω_1 . The response curve w near $\frac{\Omega L}{\pi c} = 1$ has been omitted, because at this point the solution for an infinite medium becomes meaningless (See the last paragraph but one of Section 2).

Stiffened Shells

It is worth noting that the theory developed is applicable, approximately, to ring stiffened shells provided the length L of the half wave is several times the stiffener spacing. In such a case modes of vibrations of the stiffened shell exist for which the displacements are approximately sinusoidal, so that the modes may be expressed again in the form of Eq. (1). If the frequencies and the shapes of the modes in vacuo are known* the previously derived formulae can be applied and furnish approximate results for stiffened shells. Such a treatment should give good approximations at low frequencies, while at high ones detail will be lost, because the vibrations of the shell between stiffeners have been neglected.

5. Effect of Static Pressure

The theory presented in Sections 1 and 2 neglected the effect of a static pressure in the surrounding medium on the shell.

The effect of the static pressure on the medium is indirectly allowed for by the use of an appropriate value c for the velocity of sound, but the "buckling effect" of the static pressure on the shell does not appear in the previous derivations.

Consider the shell in vacuo on which an external pressure p_0 (smaller than the

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* It is intended to treat engineering methods for the determination of these modes in a future report.

buckling one) is applied by an imaginary inertialess device. The modes of free vibrations and frequencies can be determined, furnishing a set of new modes having in general lower frequencies than in the case $p_0 = 0$. To account for the effect of the static pressure it is necessary to use these new modes and their frequencies in the previous analysis.

Provided p_0 is smaller (say at least 10%) than buckling pressure of the shell in the mode having the same half wave length L and circumferential wave number n, the shape of the modes in vacuo for zero pressure and for a pressure p_0 will not differ substantially. One can conclude that this is the case, by applying the reasoning used in ref. 1 (pp. 28, 29) to show that the shape of the modes in a thin shell will not be appreciably affected by its bending stiffness. It is therefore permissible to use the shapes of the modes without static pressure (given in ref. 1) as approximation of the new modes, but new, corrected values ω_p for the frequencies must be used. The corrected frequencies ω_p can be found from Raleigh's principle. If this is done it is found that even the frequencies change only very slightly, except the fundamental ones for $n \geqslant 1$. It is therefore sufficient to correct the fundamental frequencies for $n \geqslant 1$, and use the original ones for $n = 0$, and for the higher frequencies if $n \geqslant 1$.

The additional term in the expression for the potential energy due to a radial pressure p_0 is known from the theory of buckling (ref. 3, p. 130)

$$\frac{p_0}{2}\iint[w^2 - w_\theta^{\,2} - u_\theta^{\,2} - 2awu_z]d\theta dz \tag{55}$$

where the symbols are those of ref. 1. To be able to apply the theory for the infinitely long shell to a shell of great, but finite length it is of interest to be able to include also the effect of a longitudinal compression due to the pressure p_0 on the end surfaces. The additional contribution to the potential energy is

$$\frac{p_0}{2}\iint -\frac{a^2}{2}[w_z^{\,2} + v_z^{\,2}]d\theta dz \tag{55a}$$

Applying Raleigh's principle the corrected fundamental frequency is

$$\omega_p = \sqrt{\omega^2 - \frac{p_0}{am}B} \tag{56}$$

where, in the case of radial pressure only

$$B = \frac{(n^2 - 1)W_1^{\,2} - 2\frac{a\pi}{L}W_1U_1 + n^2U_1^{\,2}}{U_1^{\,2} + V_1^{\,2} + W_1^{\,2}} \tag{57}$$

while for combined radial and longitudinal pressures

$$B = \frac{(n^2 - 1) W_1^2 - 2\frac{a\pi}{L} W_1 U_1 + n^2 U_1^2 + \frac{1}{2} V_1^2 + \frac{1}{2} \frac{a^2 \pi^2}{L^2} W_1^2}{U_1^2 + V_1^2 + W_1^2} \tag{57a}$$

The values of B have been determined for n = 1 to 4 incl. and are listed in Table 3 for $1 \leqslant \frac{L}{a} \leqslant 10$. It will be noted that some of the values B for n = 1 are negative, indicating an increase of the frequency due to radial pressure only.

6. A Conclusion Concerning Transient Problems

The results of the analysis of forced vibrations may be utilized for the solution of transient problems. In such cases the force can be expanded in multiple Fourier series (or integrals), the individual terms of which will have the response found in Section 2.

The fact that there is a fundamental difference in the response depending on whether Ω is larger or smaller than $\frac{\pi c}{L}$, may be used to predict the character of the response of transient problems, and especially the type of approximation permissible for a particular problem.

In shock and impact problems an important part of the force will be in the range of high frequencies. The response during and shortly after the application of the force will be governed by the compressibility of the medium and radiation will be important. If on the other hand the response is to be studied after a certain time has elapsed the high frequency effects will have vanished and the elastic and inertial effects will be paramount.

Consider, for example, the case of the transverse bending stresses due to shock or impact in a submerged shell or other structure.* In order for these stresses to reach their maximum, a time comparable to the fundamental period of vibration must have elapsed. In this time the compressible effects had time to decay, and the elastic and intertial ones control. It is therefore possible to study the response with good approximation by an expansion into the modes of free vibration of the submerged structure. It should, however, be remembered that the results obtained in such a manner are not valid for the pressure in the medium or other effects shortly after the impact.

A case where the compressible effects are paramount is the response of a cylindrical shell to a transverse step shock wave treated in ref. 2. The major effect occurs in the two lowest modes within a time much too short for the compressible effects to decay. In this case the response of the shell was expressed in terms of the modes of the structure in vacuo, treating all the fluid pressures as external forces acting on the shell. The use of the modes in vacuo, and not of the modes of the submerged shell, is

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*The paper considered cylindrical shells, but it is evident that the general behavior of all types of structures must be similar.

required in cases involving infinite media where the compressibility remains important. This is due to the fact that for a shell in an infinite medium for each value of n only a finite number of frequencies and modes exist, even if complex frequencies are included. (In the above mentioned example there is for n = 0 just one frequency which is complex.) This finite number of modes is not sufficient to express the infinite number of states (for each value of n) of which the system consisting of shell and medium is capable. The number of modes available is sufficient to express any state of displacement and velocity of the shell at a given moment, but additional terms representing waves in the medium of appropriate pressure and velocity must be superimposed to express all possible simultaneous states of the medium. The use of the modes of the submerged structure is in such a case of no advantage as the additional terms mentioned remain of paramount importance. If in a shock problem sufficient time has elapsed for the effects represented by these terms to decay the solution converges asymptotically towards the modes. Because of the decay of the modes belonging to complex frequencies, only the free vibrations of real frequency will remain. This is of course nothing else but the already discussed case where compressibility is unimportant.

It can therefore be stated that, if high frequency terms occur in the force, or shock effects are wanted within a short time after the application of the force, a treatment using solely modes of vibration of the submerged structure would be incomplete, as additional terms occur in the solution.* As an alternative approach the modes of free vibration of the structure may be used as generalized coordinates describing the response of the structure fully, but leaving the medium to be treated by means of the differential equations for the potential, or in any other way desired.

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*This can also be seen by a purely mathematical approach. Obtaining the solution by a Fourier Transform, the contributions of free vibrations will appear as residues while the additional terms are branch integrals.

Table 1

Values of Δ if $\Omega \leq \frac{\pi c}{L}$

$a\beta = a\sqrt{\frac{\pi^2}{L^2} - \frac{\Omega^2}{c^2}}$	n = 0	n = 1	n = 2	n = 3	n = 4
0	∞	2.000	1.000	0.667	0.500
0.1	4.928	1.952	0.997	0.666	0.4998
0.2	3.670	1.863	0.990	0.664	0.499
0.4	2.551	1.661	0.965	0.658	0.497
0.6	1.989	1.473	0.929	0.648	0.493
0.8	1.640	1.312	0.888	0.635	0.487
1.0	1.399	1.177	0.844	0.619	0.481
1.5	1.028	0.928	0.737	0.575	0.460
2.0	0.814	0.761	0.645	0.528	0.436
3.0	0.577	0.556	0.505	0.443	0.385
4.0	0.447	0.437	0.411	0.375	0.338
5.0	0.365	0.360	0.344	0.322	0.298
6.0	0.309	0.305	0.296	0.281	0.264
7.0	0.267	0.265	0.259	0.249	0.237
8.0	0.236	0.234	0.230	0.223	0.214
9.0	0.211	0.210	0.206	0.201	0.195
10.0	0.191	0.190	0.187	0.184	0.179

Assymptotic solution for all values of n

$$\Delta = \frac{4}{2\beta a + 1}$$

Table 2

Values of Δ if $\Omega \geqslant \frac{\pi c}{L}$

$a\bar{\beta} = a\sqrt{\frac{\Omega^2}{c^2} - \frac{\pi^2}{L^2}}$	n = 0	n = 1	n = 2	n = 3	n = 4
0	∞(1 - i)	2.000	1.000	0.667	0.500
0.1	4.774 - 3.052 i	2.048 - 0.032 i	1.003 - 0.000 i	0.667 - 0.000 i	0.500 - 0.000 i
0.2	3.339 - 2.879 i	2.135 - 0.131 i	1.010 - 0.0003i	0.669 - 0.000 i	0.501 - 0.000 i
0.4	1.974 - 2.479 i	2.250 - 0.530 i	1.045 - 0.005 i	0.676 - 0.0000 i	0.503 - 0.000 i
0.6	1.297 - 2.117 i	2.092 - 1.040 i	1.107 - 0.026 i	0.688 - 0.0002 i	0.508 - 0.000 i
0.8	0.908 - 1.820 i	1.684 - 1.383 i	1.196 - 0.082 i	0.708 - 0.0014 i	0.514 - 0.000 i
1.0	0.666 - 1.584 i	1.244 - 1.478 i	1.293 - 0.199 i	0.738 - 0.0051 i	0.523 - 0.0008 i
1.5	0.354 - 1.176 i	0.570 - 1.253 i	1.271 - 0.753 i	0.858 - 0.0547 i	0.559 - 0.0018 i
2.0	0.216 - 0.925 i	0.303 - 0.988 i	0.760 - 1.025 i	1.010 - 0.270 i	0.628 - 0.016 i
2.5	0.145 - 0.759 i	0.185 - 0.801 i	0.392 - 0.913 i	0.918 - 0.655 i	0.740 - 0.086 i
3.0	0.103 - 0.642 i	0.124 - 0.669 i	0.221 - 0.759 i	0.570 - 0.812 i	0.837 - 0.292 i
4.0	0.060 - 0.489 i	0.067 - 0.502 i	0.095 - 0.546 i	0.184 - 0.632 i	0.465 - 0.684 i
5.0	0.039 - 0.394 i	0.042 - 0.401 i	0.053 - 0.425 i	0.081 - 0.471 i	0.161 - 0.550 i
10.0	0.010 - 0.199 i	0.010 - 0.200 i	0.011 - 0.203 i	0.012 - 0.209 i	0.014 - 0.217 i

For assymptotic values, See Eq. 51 & 52 on Pg. 11.

Table 3

Values of B in Eq. (56)

$\frac{L}{a}$	Radial Pressure Only				Combined Radial and Longitudinal Pressure			
	n = 1	n = 2	n = 3	n = 4	n = 1	n = 2	n = 3	n = 4
1.00	0.282	2.604	7.108	13.86	4.963	7.171	11.718	18.55
1.25	0.111	2.362	6.956	13.81	2.949	5.186	9.869	16.80
1.50	-0.0427	2.222	6.906	13.82	1.810	4.142	8.923	15.90
1.75	-0.0457	2.152	6.903	13.85	1.252	3.555	8.391	15.39
2.00	-0.212	2.126	6.922	13.89	0.763	3.207	8.071	15.07
2.25	-0.240	2.123	6.947	13.92	0.539	2.991	7.865	14.86
2.50	-0.248	2.132	6.973	13.95	0.405	2.851	7.726	14.72
2.75	-0.245	2.147	6.997	13.97	0.323	2.758	7.629	14.61
3.00	-0.236	2.165	7.019	13.99	0.273	2.693	7.559	14.53
3.50	-0.211	2.200	7.056	14.02	0.222	2.616	7.466	14.43
4.00	-0.185	2.231	7.083	14.04	0.202	2.574	7.410	14.36
5.00	-0.140	2.278	7.120	14.07	0.195	2.534	7.348	14.28
6.00	-0.107	2.310	7.143	14.08	0.201	2.518	7.316	14.24
7.00	-0.0842	2.330	7.157	14.09	0.209	2.511	7.298	14.21
8.00	-0.0672	2.345	7.166	14.10	0.215	2.507	7.286	14.20
9.00	-0.0547	2.356	7.173	14.10	0.221	2.505	7.278	14.19
10.00	-0.0452	2.364	7.178	14.10	0.225	2.503	7.273	14.18

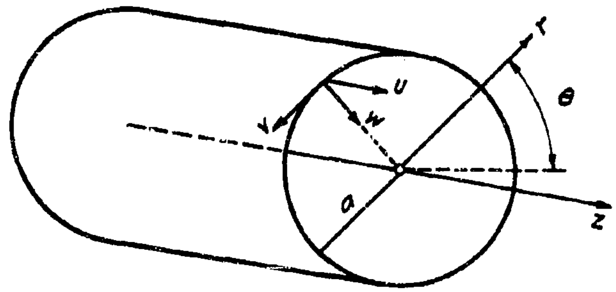

Fig. 1

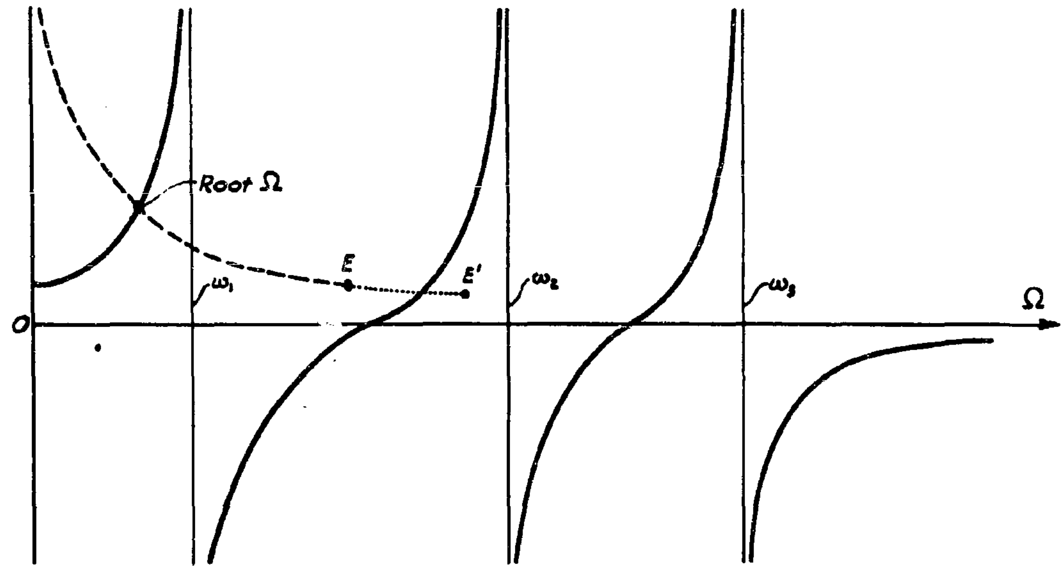

Fig. 2

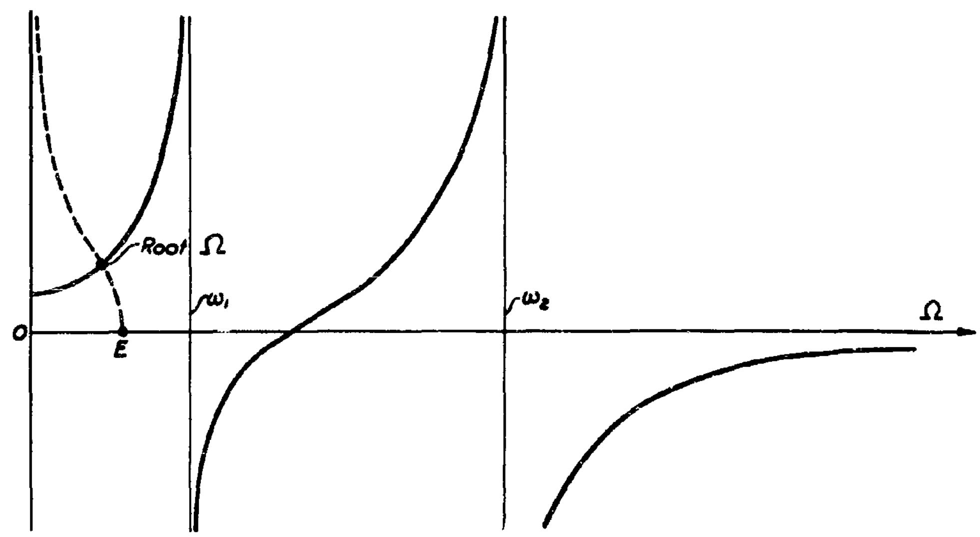

Fig. 3

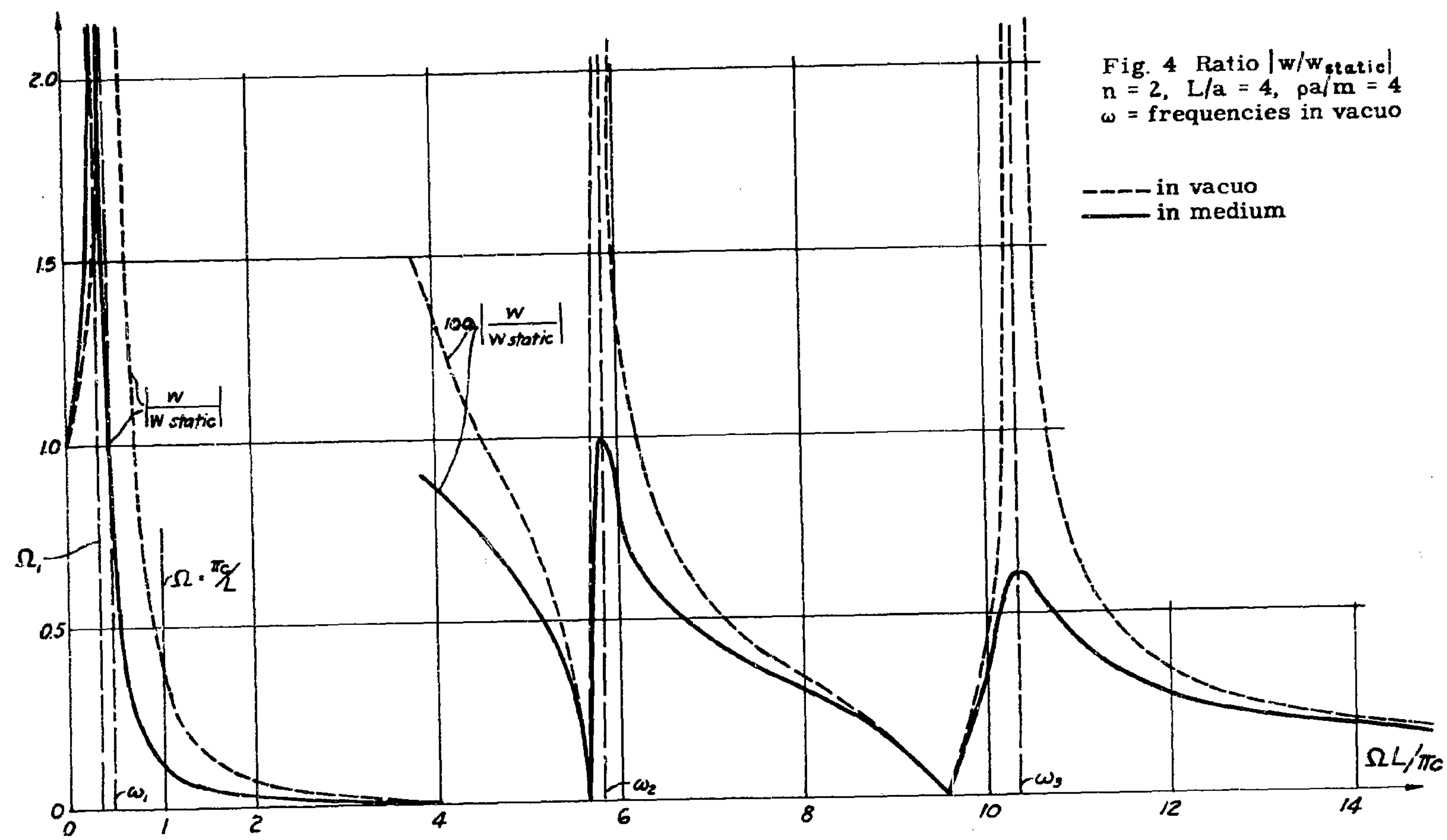

Fig. 4 Ratio $|w/w_{static}|$
$n = 2$, $L/a = 4$, $\rho a/m = 4$
ω = frequencies in vacuo

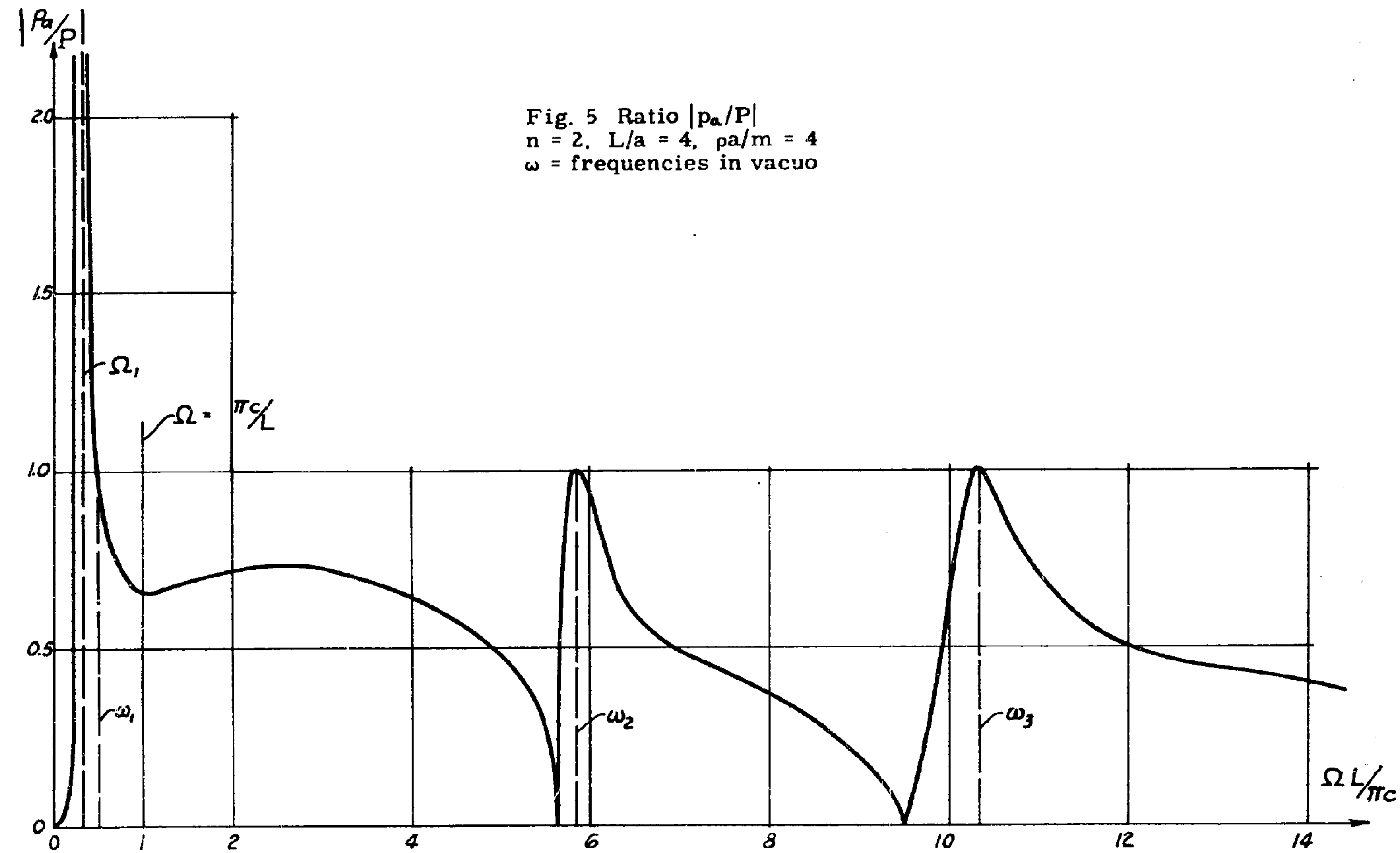

Fig. 5 Ratio $|p_a/P|$
$n = 2$, $L/a = 4$, $\rho a/m = 4$
ω = frequencies in vacuo

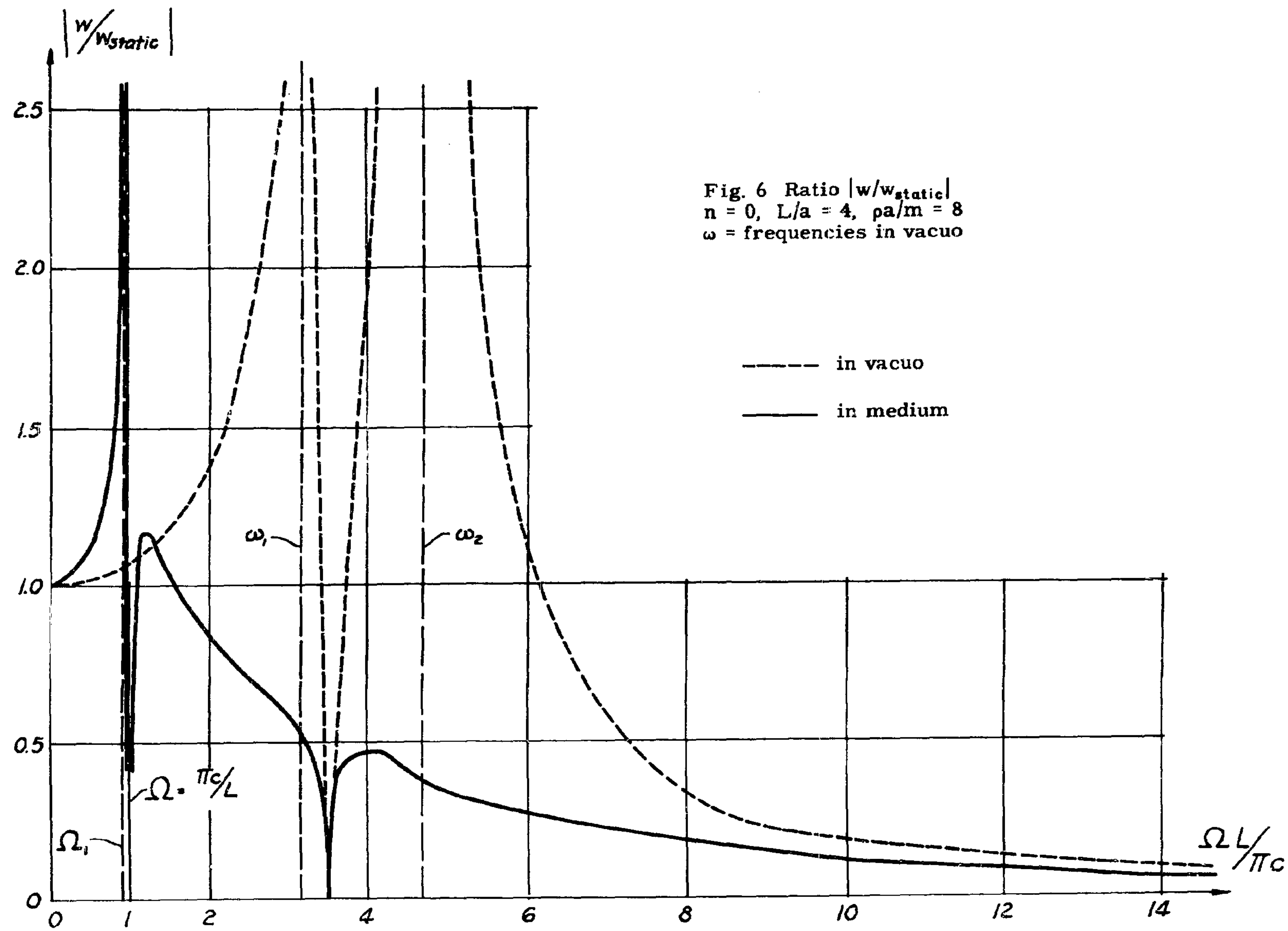

Fig. 6 Ratio $|w/w_{static}|$
$n = 0$, $L/a = 4$, $\rho a/m = 8$
ω = frequencies in vacuo

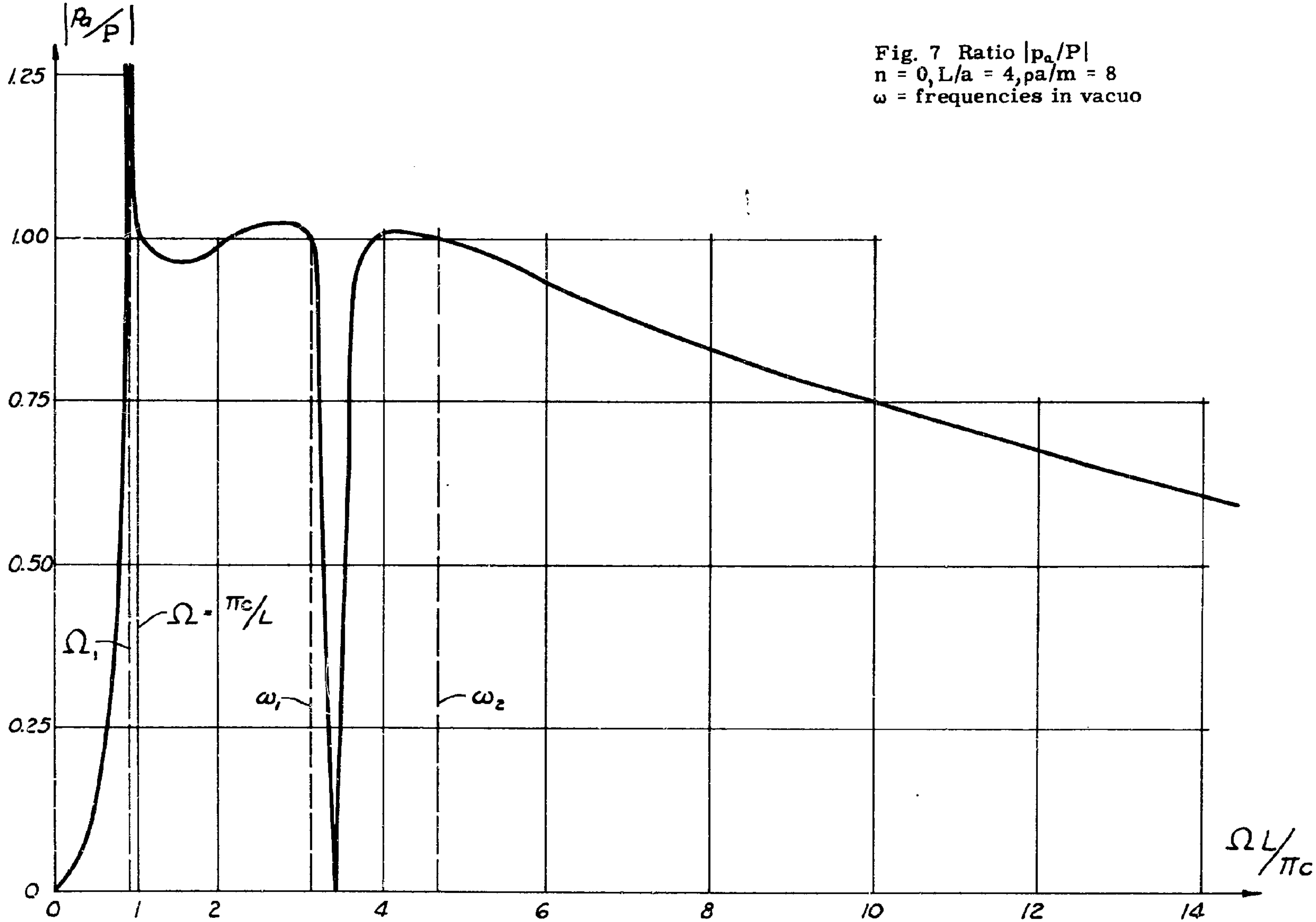

Fig. 7 Ratio $|p_a/P|$
$n = 0, L/a = 4, \rho a/m = 8$
ω = frequencies in vacuo