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Analysis of the Hydrogen Eclipse Observations to
Determine the Thermodynamic State of
the Solar Chromosphere

by

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FOREWORD
(By Walter Orr Roberts)

The following report was prepared by the authors as a part of their
work in preparation for the eclipse expedition of High Altitude Observa-
tory to the Sudan on 25 February 1952. It was written before the eclipse.
We issue it now as we prepare to use the methods outlined for reduction
of the spectra obtained. We express our thanks to the authors and to the
University of Utah for the many forms of assistance they have provided
to us in this work. Without this aid we would have found it impossible
to make sound plans for the expedition in the short time that was
available to us.

* * * * * *

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I. Introduction

The following discussion contains an outline of, and a justifi-
cation for, the theoretical analysis of the data to be obtained at
the 1952 eclipse. Actually, the following report considers only a
limited portion of the expected data -- that relating to the hydrogen
Balmer and Paschen series, and the Balmer continuum. An analysis [1]
of earlier eclipse observations [2] demonstrated the feasibility of
such an analysis. On the basis of the results from this analysis,
we suggested the advisability of an eclipse expedition, with limited
but specific objectives, aimed only at the data required for this
analysis. We proposed deliberately foregoing the usual attempts to
measure all possible phenomena associated with the eclipse. The
1952 eclipse program, undertaken by High Altitude Observatory in
close cooperation with and supported by the Naval Research Laboratory,
had somewhat wider objectives, but the effort was concentrated on the

objectives outlined here. Dr. John P. Hagen and his associates
planned to conduct simultaneous radio observations at two frequencies
during the eclipse. It was hoped thereby to obtain some resolution
of the current dilemma between radio and optical observations of the
outer solar atmosphere. The optical observations were to be made by
High Altitude Observatory, and the analysis to be carried out in
close cooperation with the Utah group. The radio observations were
to be handled by the Naval Research Laboratory group. The following
discussion relates only to the analysis of the hydrogen observations
originally proposed.

The general philosophy of the program to be discussed is one of
an attempt to investigate certain gross features of the hydrogen
spectrum that may be unambiguously interpreted to specify certain
gross features of the structure of the chromosphere. Since the solar
chromosphere presents so many observational anomalies, such attempts
may be overly optimistic. Nonetheless, by restricting ourselves to
hydrogen we may hope, because of its preponderant abundance, to
establish results characterizing the thermodynamic structure of the
chromosphere. Furthermore, by considering only relative spectral
intensities, we may hope to avoid the usual difficulties encountered
with absolute standards. If the procedure is successful in establish-
ing these gross features, we may then proceed to analyze the full
details of the spectrum to obtain the full details of the state of
the chromosphere. There is a certain amount of risk in attempting to
so divide the problem of the state of the chromosphere -- one recalls
attempts to predict the temperature of the planetary nebulae from the
action of the radiation field of the exciting star on the hydrogen
alone without considering the thermostat action of the oxygen
"impurities." It is thus essential to examine critically all data,
not only that for hydrogen, seeking contradictions to the present
procedure. The difficulty in such examinations -- and several have
been offered as, for example, the criticism of the high temperature
model because of the lack of metallic metastable lines -- lies in an
inability to evaluate the exact situation to be expected until the
gross thermodynamic structure is known. We attempt then to obtain
this gross structure, selecting methods general enough to remove the
above possibilities of internal inconsistency.

II. Method of Reduction.

A. The Balmer and Paschen decrements.

Specifically, one observes during an eclipse all the light
originating above a certain height in the chromosphere. Let us
consider, then, the energy emitted by transitions from the nth
to the second level of the hydrogen atom in terms of the state
of the chromosphere. We consider one cm^3 of matter at (x, y)
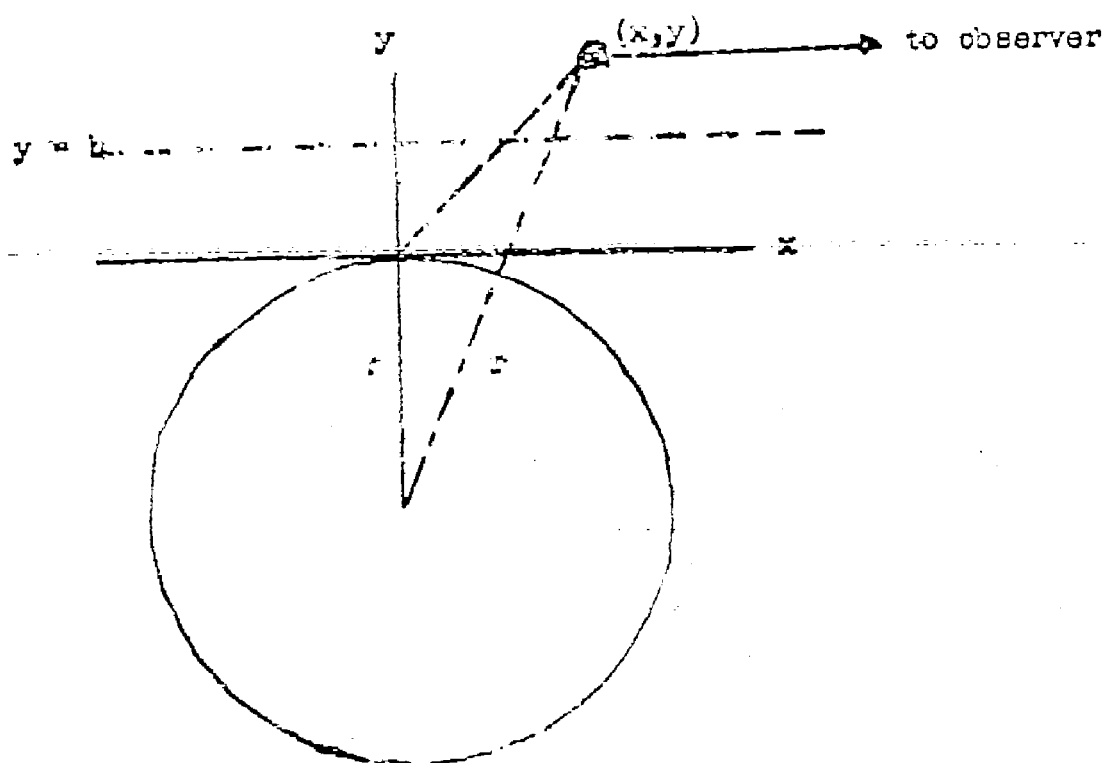in the chromosphere.

Figure I

w is the distance from (x, y) to the surface of the sun. We shall be specifically interested in ascertaining what characteristics of the thermodynamic state can be varied to give agreement between the theoretical and the observed Balmer decrement. One finds that on the basis of thermodynamic equilibrium and an atmosphere transparent to its own radiation, the computed decrement is in every case too high. Thus let us compute the energy per unit solid angle E_{n2} with all possible variations of the above, evidently over simplified, assumptions included. We include, therefore, departures from thermodynamic equilibrium by introducing the standard b_n defined in equation (6) below, include self-absorption effects, and make no attempt to specify the electron temperature, T_e. Thus, in standard notation,

$$d_c E_{n2} = N_n(x,y) h \nu_{n2} A_{n2} \frac{1}{4\pi} dV \qquad (1)$$

where N_n is the number of particles/cm^3 at x,y and we are considering the radiation from the element of volume dV. It will be easiest if we consider the radiation from a slab of width 1cm in

the z direction. Thus we operate with $\int_{z_0 - \frac{1}{2}}^{z_0 + \frac{1}{2}} dz$ on (1) and we obtain

$$d_0 E_{n2} = \frac{N_n(x,y) h \gamma_{n2} A_{n2}}{4 \pi} \, dxdy \tag{2}$$

The phenomenon of self absorption, of course, decreases this energy so that only the amount

$$dE_{n2} = d_0 E_{n2} \cdot \exp\left\{ -a_\gamma \int_y^\infty N_2(x,y) \, dx \right\} \tag{3}$$

finally emerges from the atmosphere. We write

$$N(x,y) = \int_y^\infty N_2(n,y) dx \tag{4}$$

so that

$$dE_{n2} = d_0 E_{n2} \cdot e^{-a_2 N(x,y)} \tag{5}$$

Now, using $X_n = \frac{\chi_n}{k^2 \epsilon}$ where χ_n is the energy to ionize from the nth level (assuming hydrogen sufficiently ionized that $N_1 = N_\epsilon$):

$$N_n = N_\epsilon^2 \, b_n \left(\frac{h^2}{2 \pi m k T_\epsilon} \right)^{3/2} \frac{\varpi_n}{2} \, e^{X_n} \tag{6}$$

Thus the total energy emitted is

$$E_{n2} = \iint d E_{n2}$$
$$= \frac{h \gamma_{n2} A_{n2}}{4 \pi} \left(\frac{h^2}{2 \pi m k} \right)^{3/2} \frac{\varpi_n}{2} \int_h \int_{-\infty}^{+\infty} b_n N_\epsilon^2 T_\epsilon^{-3/2} e^{X_n} \, e^{-a_2 N(x,y)} dxdy$$

$$E_{n2} = C_1 \gamma_{n2} A_{n2} \varpi_n \int_h \int_{-\infty}^{+\infty} b_n N_\epsilon^2 T_\epsilon^{-3/2} e^{X_n} \, e^{-a_2 N(x,y)} dxdy \tag{7}$$

where C_1 is a numerical constant. The integral may be thought of as a weighted average of the emission from -- hence the population of -- the nth level. We find it convenient therefore to also use in the discussion equation (7) without the substitution (6):

$$E_{n2} = \frac{A_{n2} \, h \gamma}{4 \pi} \int_h \int_{-\infty}^\infty N_n(x,y) e^{-a_2 N(x,y)} \, dxdy \tag{8}$$

Consider now equation (7). The Balmer decrement corresponds to a differentiation, or differencing, of equation (7) with respect to the quantum number n. Performing this operation we obtain:

$$\Delta_n \frac{E_{n2}}{C_1} = \left(\ldots\right) \int\int \left[\ldots\right] \left\{ \frac{\Delta_n b_n}{b_n} + \frac{2Xn}{n} - \frac{3!a}{n} \right\} \Delta_n \left(\omega_n V_{n2} \Delta n2\right) \int\int \left[\ldots\right]$$

$$\underset{0}{\vee} \qquad\qquad \underset{0}{\vee} \quad \underset{0}{\vee} \quad \underset{0}{\wedge} \qquad \underset{0}{\vee}$$

$$(9)$$

$$\Delta_n \longrightarrow f(n) - f(n + k)$$

where the inequality sign refers to the whole term, including the accompanying sign. We have also taken the absorption coefficient a_{n2} to vary as n^{-3}. The dotted brackets in each case contain a positive quantity. The result on the b_n comes only from the assumption that $T_e > T_{eff}$ for the sun — and independent of the numerical value of T_e. We see thus that only the third term gives a negative contribution to the right hand side. Thus only the self-absorption effect can remedy the difficulty mentioned above — that the observed Balmer decrement is smaller than the thermodynamic equilibrium value. The effect of departure from thermodynamic equilibrium only increases the discrepancy, unless one wishes to adopt $T_e < T_{eff}$, a hardly plausible result these days. Furthermore, we shall in the following section (III A 1) demonstrate from the observations that $b_n > b_{n+k}$.

We recognize now that the self-absorption is a function of one parameter only, the population of the two-quantum level. Hence we should be able to use the observed Balmer decrement in conjunction with the result from equation (9) to obtain this population of the two-quantum level as a function of height. Since we do not a priori know the thermodynamic structure of the atmosphere, the procedure must necessarily be one of successive approximation. In the analysis earlier mentioned,[1] only the first approximation was carried out, i.e., an isothermal chromosphere was assumed and departures from thermodynamic equilibrium neglected. Actually, we note from equation (8) that most of the emission comes from the lower, central part of the atmospheric region observed; so the first approximation is fairly accurate. The plans for the forthcoming eclipses indicate that considerably more observational data will result, so the successive approximation scheme will be necessary to analyze the data.

Consider the first approximation. For constant T_e, and $b_n \cong 1$, we have an exponential gradient for N_e which we may write:

$$N_e^2 = N_{eo}^2 \; e^{-\beta x - \frac{\beta y^2}{2r}} \tag{10}$$

according to the usual geometric approximation for $y \ll r$. The contribution of large y to the chromospheric observations is negligible. We have now two alternative procedures to carry out the computation.

In the first, we substitute (10) in both equation (4) and equation (7), and carry out the integration in series form. We obtain.[1]

$$E_{n2} = \left(C_2 T_e^{-\frac{3}{2}} N_{eo}^2 \frac{V}{\lambda} \widetilde{\omega}_n \nu_{n2} A_{n2} \right) e^{X_n} \left[1 + \sum_{m1} (-1)^m \left(\bar{N}_2 a_{n2} \right)^m (m+1)^{-\frac{3}{2}} \right.$$
$$\left. \left(1 + \frac{M1}{2r(m-2)!} I_2 + \dots \right) \right] \tag{11}$$

where $\bar{N}_2$ is the number of second-quantum level atoms along the line of sight and:

$$I_j = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} g^j(s^2) ds \qquad g(s^2) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-s^2} ds \tag{12}$$

C_2 is a constant.

The expression (11) represents the thermodynamic equilibrium expression save for the last bracket, which represents the self-absorption. The expression is convenient to use for small absorption (small $\bar{N}_2 a_{n2}$) -- it does not converge for large $\bar{N}_2 a_{n2}$ values. For the higher Balmer lines, however, satisfactory values for $N_2 a_{n2}$ may be quickly obtained by:

$$\log \left[\frac{1 + \sum \dots (\bar{N}_2 a_{n2})}{1 + \sum \dots (\bar{N}_2 a_{(n+1)2}} \right] = \text{observed decrement} - \Delta_n \log \left[\widetilde{\omega}_n \nu_{n2} A_{n2} \right] \tag{13}$$

We have necessarily omitted the term e^{X_n}, since it depends upon the (unknown) value of T_e. We note however that the omission is trivial for high enough $\underline{n}$ - and we prefer the later Balmer lines anyway to avoid convergence difficulty with the series expression for the absorption. The left side of equation (13) may be reduced to a function of a single variable by reducing $a_{(n+k)2}$ to a_{n2} using an n^{-3} dependence for a_{n2}. This procedure was followed in reference (1).

Alternatively we may proceed from equation (8). Recognizing that we may write the emission coefficient as:

$$j_\nu = A_{n2} h\nu \, N_n \tag{14}$$

and the absorption coefficient:

$$k_\nu = a_{n2} N_2 \tag{15}$$

with the associated tangential optical depth:

$$d\mathcal{T}_\nu = a_{n2} N_2 dx \tag{16}$$

we may write (8):

$$E_{n2} = \frac{1}{4\pi} \int_h^\infty \int_o^{\mathcal{T}} \frac{j_\nu}{k_\nu} \, e^{-\mathcal{T}} \, d\mathcal{T} \, dy \tag{17}$$

We note now the expression (15) must include induced emissions to be complete - so that we may write:

$$a_{n2} = a'_{n2} \left(1 - \frac{N_n}{N_2} \frac{\varpi_2}{\varpi_n} \right) \tag{18}$$

and use the relation:

$$\frac{A_{n2}}{a'_{n2}} = \frac{8\pi\nu^2}{c^2} \frac{\varpi_2}{\varpi_n} \tag{19}$$

whence:

$$\frac{j_\nu}{k_\nu} = \frac{2h\nu^3}{c^2} \frac{\varpi_2}{\varpi_n} \frac{N_n}{N_2} \left[1 - \frac{N_n}{N_2} \frac{\varpi_2}{\varpi_n} \right]^{-1}$$

$$= \frac{2h\nu^3}{c^3} \left[\frac{b_2}{b_n} e^{h\nu/kT_2} - 1 \right]^{-1} \tag{20}$$

using equation (6).

Alternatively then we may write (17) as:

$$E_{n2} = \frac{2h\nu^3}{c^2} \int_n^\infty \int_0^T \left[\frac{b_2}{b_n} e^{h\nu/kT_e} - 1 \right]^{-1} e^{-T} \, dT\,dx \qquad (17.1)$$

We write (17) in both forms to indicate the nature of the approximation made. In the usual discussion, j_ν/k_ν is taken as function of the temperature only - in which case we could remove it from under the integral in (17). Actually, however, the ratio depends upon N_e as well as T_e . In discussing the first method of analysis we have taken $b_n = 1$. There, however, we were concerned only with the higher n values. A variation in b_2 entered as a second order correction through the integral of the exponential. In (17.1), however, b_2 enters directly. The approximation that $b_n \sim 1$ for large n is much better than that $b_2 \sim 1$. Thus, combining these two results - the first order appearance of b_2 and the likelihood of its considerable departure from unity, it does not seem too satisfactory an approximation to remove j_ν/k_ν from the integral. We note, however, that we shall be differencing the result - and so some of the error will disappear. Thus we proceed on this approximation and obtain:

$$E_{n2} = \frac{2h\nu^3}{c^2} \left[\frac{b_n}{b_2} e^{h\nu/kT_e} - 1 \right]^{-1} \int_h^\infty \left(1 - e^{-T(x)} \right) dx \qquad (21)$$

We note that this expression complements that obtained by the first method - for as T becomes sufficiently great the emission per unit area - i.e., ignoring the last integration over the atmosphere lying above a given point --approaches the Planck function modified by the b_n/b_2 factor. Thus we have an expression suitable for the case of high absorption, when the series expansion in equation (11) fails to converge.

The actual evaluation of the integral in (21) must be performed numerically - for we note that $T(x)$ is the $N_2 a_{2n}$ of the previous method and decreases exponentially with height. For small T, the exponential may be expanded and the integral evaluated as:

$$\int_h^\infty \left[1 - e^{-\bar{N}_2(x)a} \right] dx = \left(\bar{N}_2(h)a - \frac{[\bar{N}_2(h)a]^2}{2 \cdot 2!} + \frac{[\bar{N}_2(h)a]^3}{3 \cdot 3!} \right)(1 - e^{-\bar{N}_2(h)a}) \quad (22)$$

so that this expansion may be used if preferred.

The last procedure discussed is actually of more value in the second and higher approximations, where some idea of the variation of j_ν/k_ν with T is had. It is largely for this reason that we have discussed it. If desired however, equations (21) and (22) may be used in place of equation (11).

From the above procedures, then, we obtain first approximation values of $N_2(h)$ from the Balmer decrement. Clearly the discussion may be taken over intact to observations of the Paschen decrement, obtaining a check on the N_2 values. For the higher order approximations, we require a knowledge of the N_e and T_e values in order that the isothermal and $b_n = 1$ assumptions may be dropped. Clearly, some information on these quantities should result from the knowledge just gained of $N_2(h)$. There are, in addition, several other sources of information. We turn to consider them.

B. Determination of N_e and T_e.

We have available three distinct methods of estimating various ones of the unknown quantus N_e, $\nabla_h N_e$, T_e, $\nabla_h T_e$ from the eclipse optical observations. One method represents an application and extension of the results of Section A. Because it is an extension, and because it was developed simultaneously with that method, we consider it first. The method gives information on T_e, $\nabla_h T_e$, and $\nabla_h \ell n T_e$. The second method rests on a study of line profiles and gives information on T_e and N_e at each height for which precise observations exist. The third, and possibly least accurate method, leads to values of T_e.

1. The apparent emission height-gradient.

Consider two methods commonly used in obtaining the height gradient of hydrogen in the solar atmosphere. In the first method we compare the emission from a given Balmer line at two eclipse heights. With the exception of an error resulting from the neglect of self-absorption effects, the emission height-gradient may be interpreted directly as the height gradient of the number of atoms in the upper level of the line. The neglected self-absorption effects clearly make this inferred value a lower limit to the true value. In the second method we attempt to eliminate self-absorption affects by observing the height at which the various Balmer lines just vanish. Presumably, the various lines reach the same intensity at their respective vanishing points, and the emission is so small that self-absorption affects drop out. By correcting for the difference in transition probability, the comparison of vanishing points for the two lines gives the ratio of the population of the two levels at the two heights. The problem is to convert this population ratio into a height gradient for the population of either level. From equation (6) we see that, if the b_n values were the same for the two levels at a given height, the figure just obtained would, when corrected for the Boltzmann factor exp (X_n), give the height gradient of either level. If,

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however, $b_n > b_{n+k}$, then the emission at a given height
for Balmer line H_n would be greater than it would be in
the case of $b_n = b_{n+k}$; the line H_n would persist to a
greater height relative to H_{n+k}; and we would underestimate
the height gradient if we used the observational material
with no b_n correction. If $b_n < b_{n+k}$, the reverse situation
would hold. We can then compare the density gradients
obtained by the first and second methods, in the hope of
obtaining the behavior of b_n. We note, of course, that a
result showing the density gradient from the first method
to be less than that from the second would be somewhat
inconclusive because of the neglected self-absorption
effect. Fortunately, however, we find the first-method
gradient to exceed the second. The situation is not
peculiar to data from one eclipse, nor is the difference
so small as to be masked by observational error. (The
difference is almost a factor of 2.) Hence it would appear
that, from this comparison of two methods of estimating
the hydrogen density gradient, we have a conclusive
demonstration of the direction of deviation of the chromo-
sphere from thermodynamic equilibrium; that is
$b_n > b_{n+k} > 1$.

Thus, we have demonstrated $b_n > b_{n+k}$, the results needed
for the Section A analysis of self-absorption. We proceed
now to make further use of the apparent emission gradient.
We have remarked that the apparent emission gradient differs
from the true density gradient by the self-absorption effect.
We can, however, correct for this self-absorption by using
the results from Section A. Then the corrected results give
the true height gradient of N_n, and we can write for each
N_n, $n > 2$, at each height, from equation (6):

$$\frac{d \ln N_n}{dh} = \frac{d \ln N_e^2}{dh} - \frac{d \ln T_e}{dh}\left[\frac{3}{2} + \chi_n\right] + \frac{d \ln b_n}{dh} \qquad (23)$$

where the left side is known from the above corrected emission
gradient. We can also write this equation for N_2, using the
results from Section A to obtain the left side. Thus we have a
set of equations, as many as we have observed hydrogen lines,
Balmer or Paschen, with the unknowns $\nabla_h \ln N_e$, $\nabla_h T_e$, T_e,
and the $\nabla_h \ln b_n$. While the $\nabla_h \ln b_n$ are obviously not all
independent, the dependence cannot be computed until N_e, T_e
are known. Thus a preliminary solution of the equation
(23) must be carried through ignoring the b_n term. The
error is greatest in the b_2 term, the N_2 equation, so it
is preferable to carry through a solution neglecting the
N_2 equation if possible. Then the second approximation
to both this procedure and that of Section A may be made.

We note, however, that neither this Section B 1, nor Section A provide values of N_e directly. One can estimate a value by successive approximations on a consistency basis, but it is useful to have more direct methods in the first approximation.

Before proceeding to the second method we note that the preceding discussion enables a test of the validity of the hydrostatic equilibrium to be made. We can write this equation as:

$$0 = \frac{d}{dh} \ln \frac{N_e^2}{(1-\eta)^2} + 2 \frac{d\ln T_e}{dh} + 2 \frac{\mu M_H}{RT_e} g \qquad (24)$$

where η is the fraction of the hydrogen yet unionized. Once estimates of T_e and N_e are available, η can be estimated. The remaining quantities in (24) are known, and the validity of the equation can be checked.

2. Results from line profiles.

There are two varieties of measures involving line profiles. The first is a direct measure of a single line profile. This technique has been used by R. V. Redman,[3] who finds the early hydrogen lines and the the metallic lines exhibit Doppler profiles. Redman plans to repeat these measures at the 1952 eclipse. No such work is contemplated in our plans. We shall, therefore, not consider this method further. It is not meant thereby to underemphasize the method, for on the contrary it is the most reliable method for clearly indicating the kinetic state of the atmosphere. Thus the method is critical in any discussion of atmospheric kinetic temperature.

The second method dealing with line profiles usually concerns the wings of the lines, where the effect of collisional broadening becomes of importance. We note, however, that for sufficiently high temperature the thermal broadening can influence appreciably the merging of the lines. We consider the problem.

The Stark broadening due to the positive ions causes a merging of the lines at finite n. An approximate relation, agreeing well with experiment, between the n value for the last resolved line and the ion density has been computed by Inglis and Teller:[4]

$$\log N_i = 23.26 - 7.5 \log n \qquad (25)$$

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The relation considers, however, only Stark broadening, and we wish also to include the Doppler broadening, given by the expression:

$$e^{-\left(\frac{\nu-\nu_0}{\nu_0}\cdot\frac{c}{v}\right)^2} \tag{26}$$

Now, the line profile for the Stark effect will be of the form

$$\frac{a^2}{a^2+(\nu-\nu_0)^2} \tag{27}$$

Thus, the Doppler and Stark effect together will yield

$$a^2 \int_{-\infty}^{+\infty} \frac{e^{-\xi^2\left(\frac{c}{\nu_0 v}\right)^2}}{a^2+(\xi-\nu)^2}\, d\xi \tag{28}$$

Unfortunately the result does not come out explicitly in any neat form. Therefore, noting that the results to be obtained are only approximate anyway because of the uncertainty as to when a line is exactly resolved or unresolved, we use a somewhat approximate method and thereby save much labor. We wonder what gaussian curve will approximate to (27) so far as resolution criteria are concerned. Clearly a is the width at half maximum in (27). We should like to replace (27) by

$$e^{-\left(\frac{\nu-\nu_0}{\alpha}\right)^2} \tag{29}$$

and find α in terms of a. We shall adjust α so as to give the same resolution as (27).

We shall use the resolution criterion that for a gaussian distribution, the half-width at half-maximum is just one half $\Delta\nu$ the line separation. Thus, for resolution,

$$\alpha = \frac{\Delta\nu}{2\sqrt{\ln 2}} \tag{30}$$

We see then that midway between the lines the intensity is

$$I_m = 2\cdot\frac{1}{2} = 1$$

and at the center

$$I_c = 1 + e^{-4\ln 2} \tag{31}$$

Thus

$$\frac{I_c-I_m}{I_m} = \delta\ln I = e^{-4\ln 2} \tag{32}$$

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We now ask ourselves how a is related to α so that $\delta \ln I$ will have the same value when using (27) as when (29) is substituted for (27).

$$\frac{I_c - I_m}{I_m} = \frac{\left(1 + \frac{a^2}{a^2 + (\Delta \nu)^2}\right) - 2 \frac{4a^2}{4a^2 + (\Delta \nu)^2}}{\frac{6a^2}{4a^2 + (\Delta \nu)^2}}$$

One finds, upon setting this equal to $\delta \ln I$ (which is 1/16) that

$$a^2 = \Delta \nu^2 \left(\frac{1}{2} + \alpha^2 (\delta \ln I)\right)$$

or   (33)

$$a^2 = \frac{\Delta \nu}{2} = \alpha \sqrt{2 \ln 2}$$

Thus, we see that as far as resolution is concerned, a dispersion profile with a half width at half maximum of a, may be replaced by a gaussian with mean width $a/\sqrt{2 \ln 2}$ or a half width at half maximum of $a/\sqrt{2}$. Therefore, we write the combined Doppler and Stark profile as

$$e = \frac{(\nu - \nu_0)^2}{\left(\nu_0 \frac{v}{c}\right)^2 + \frac{a^2}{2 \ln 2}}$$ (34)

where a is the half width at half maximum of the Stark profile.

Now a is independent of the ground state since it is only the higher states which are significantly perturbed by the Stark effect. Thus, for a given upper level n, the energy perturbation is ΔE_n. And the resulting frequency spread is

$$\Delta \nu_n = \frac{1}{h} \Delta E_n$$

We see then that the density broadening will be the same for the Balmer and Paschen series. We consider now the Doppler broadening.

$$\nu_{nm} = \frac{R}{h} \left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$ (35)

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Adjacent lines are separated by

$$\nu_{n\,n+1} = \frac{R}{n}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{R}{n}\ \frac{2n+1}{n^2(n+1)^2} \tag{36}$$

If we take a line as unresolved when it just falls to e^{-k^2} its maximum value for the frequency of the center of the next higher line, then for the two series with the ground states m and m' we have:

$$k^2 = \frac{\nu_{n\,n+1}^2}{\left(\dfrac{\nu_{nm}\,\nu}{c}\right)^2 \ \left(\dfrac{\Delta\nu_{n'}}{2\,\ell n\,2}\right)^2} \tag{37}$$

$$k^2 = \frac{\nu_{n'\,n'+1}^2}{\left(\dfrac{\nu_{n'm'}\,\nu}{c}\right)^2 + \left(\dfrac{\Delta\nu_{n'}}{2\,\ell n\,2}\right)^2} \tag{38}$$

where n' −1 and n − 1 are the last resolved lines of the m and m' series respectively. The subscripts in each case indicate the quantum numbers involved in the quantities. We use the results of Inglis and Teller to compute $\Delta\nu_n$. It should be noted that we cannot simply use their final results because, while we have merging of the lines as did they, part of our merging is a Doppler broadening. Thus, we wish to compute $\Delta\nu_n$ for an unmerged line. They have that for a field F, the energy spread in a level is

$$\tfrac{3}{2}\ ae\ F$$

a is the radius of the orbit and is $n^2 a_0$ where a_0 is the radius of the first Bohr orbit. Thus, the frequency spread is

$$\Delta\nu = \frac{3aeF}{2h}$$

In terms of N the number of singly charged ions present Holtzmark [5] gives

$$F = 3.7\ e\ N^{2/3}$$

Thus

$$\Delta\nu = \frac{5.6\ n^2 e^2\ a_0\ N^{2/3}}{h} \tag{39}$$

or $\quad \Delta \nu = n^2 \mathcal{H} N^{2/3}$ $\qquad\qquad$ (40)

where $\quad \mathcal{H} = \dfrac{5.6\, e^2 a}{h}$ $\qquad\qquad$ (41)

Thus, using (40) we rewrite (37) and (38) as

$$\nu^2_{nm}\left(\frac{v}{c}\right)^2 + n^4 \frac{\mathcal{H}^2}{2\,\ell n\,2} N^{4/3} = \frac{\nu^2_{n,n+1}}{k^2} \qquad (42)$$

$$\nu^2_{n'm'}\left(\frac{v}{c}\right)^2 + n'^4 \frac{\mathcal{H}^2}{2\,\ell n\,2} N^{4/3} \quad \nu'^2 \frac{n'n'+1}{k^2}$$

We wish to solve for $\frac{v}{c}$ and N. We find

$$\frac{v}{c} = \frac{1}{k}\left(\frac{\left(\frac{n!}{n}\right)^4 \left(\frac{2n+1}{n+1}\right)^2 - \left(\frac{n}{n'}\right)^4 \frac{(2n'+1)^2}{(n'+1)^4}}{\left(\frac{n!}{n}\right)^4 \left(1-\frac{n^2}{n^2}\right)^2 - \frac{n^4}{n'^4}\left(1-\frac{n'^2}{n^2}\right)^2}\right)^{\frac{1}{2}} \qquad (43)$$

$$N = \left(\frac{R^2\, 2\,\ell n\,2 \left(\frac{2n'+1}{n'^2(n'+1)}\right)^2 \frac{1}{n^4}\left(1-\frac{n^2}{n^2}\right)^2 - \left(\frac{2n+1}{n^2(n+1)}\right)^2 \frac{1}{n'^4}\left(1-\frac{n'^2}{n'^2}\right)^2}{h^2\, k^2\, \mathcal{H}^2 \left(\frac{n!}{n}\right)^4 \left(1-\frac{n^2}{n^2}\right)^2 - \left(\frac{n}{n'}\right)^4 \left(1-\frac{n'^2}{n'^2}\right)^2}\right)^{3/4} \qquad (44)$$

or

$$\frac{v}{c} \sim \frac{1}{k}\left(\frac{n'^4}{n^6}\,\frac{\left[1-\left(\frac{n}{n'}\right)^{10}\right]}{\left[1-\left(\frac{n}{n'}\,\frac{n}{n'}\right)^4\right]}\right)^{\frac{1}{2}} \qquad (43.1)$$

$$N \sim \left\{\frac{2R^2\,\ell n\,2}{k^2\,\mathcal{H}^2\,h^2}\,\frac{4}{n'^{10}}\,\frac{\left[1-\left(\frac{n'}{n}\right)^6\left(\frac{n}{n'}\right)^4\right]}{\left[1-\left(\frac{n}{n'}\,\frac{n}{n'}\right)^4\right]}\right\} \qquad (44.1)$$

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If we use the resulting criterion cited above that the full width at half maximum is equal to $\gamma_{h, n+1}$, then

$$e^{-k^2/4} = 1/2$$

$$k = 2\sqrt{\ln 2} = 1.664 \tag{45}$$

Further $\mathcal{K} = 1.033$ cm^2/sec. Thus, given the last resolvable lines of Balmer and Paschen series, values for N_e and T_e result. (The value of T_e follows directly from the value of v/c.) We note, incidentally, that the expressions (43.1) and (44.1) each fall into two parts, the second part in each being a correction factor of the order of unity involving each of the n, n', m, m' values. The other factor (43.1) involves only the Balmer series data - n and m - and is the expression that would be obtained were only thermal broadening included in the discussion. Thus, if the second factor were sufficiently near unity, the T_e value would be fixed by the Balmer series alone. Similarly, the first factor in (44.1) depends only upon the Paschen series. We note, however, that the first factor is not the same as would be obtained by considering Stark broadening alone, but differs by $2^{-3/4}$. This factor has been introduced by the process of representing the Stark broadening in gaussian form. Had the half width at half-maximum for the gaussian representation been chosen the same as the half-width in the conventional expression (27), i.e., a and not $a/\sqrt{2}$, the extra $2^{-3/4}$ factor would not appear in (44.1).

3. Measures in the continuum.

In the past, attempts have been made to compute the electron temperature by using the ratio of intensities at two frequencies in the Balmer continuum.[6] However, considerable doubt is cast on these calculations by their failure to take into account the scattering by the electrons of the photospheric radiation. If N_e is the electron density then the intensity at a given frequency due to both bound-free emission and scattering is

$$E_\gamma = k_1 N_e^2 \, e^{-\frac{h\nu}{kT_e}} + k_2 N_e \, e^{-\frac{h\nu}{kT_\gamma}} \, \nu^3 \tag{46}$$

neglecting the small self-absorption effects.[7] $\nu^3 e^{-\frac{h\nu}{kT_\gamma}}$ represents the black body radiation from the photosphere of temperature T_γ. If we take the ratio of E_2 for the two frequencies ν and ν', then

$$\frac{E_\gamma}{E_{\gamma'}} = \frac{k_3 N_e \, e^{-\frac{h\nu}{kT_e}} + \nu^3 \, e^{-\frac{h\nu}{kTr}}}{k_3 N_e \, e^{-\frac{h\nu}{kT_e}} + \nu'^3 \, e^{-\frac{h\nu'}{kTr}}} \tag{47}$$

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We see that

$$\frac{E_\nu}{E_{\nu'}} \sim e^{\frac{h}{kT_e}(\nu' - \nu)} \qquad \text{for large } N_e \qquad (48)$$

and

$$\frac{E_\nu}{E_{\nu'}} \sim \frac{\nu^3}{\nu'^3} \frac{e^{-\frac{h\nu}{kT_{\nu'}}}}{e^{-\frac{h\nu'}{kT_{\nu'}}}} \qquad \text{for small } N_e \qquad (49)$$

Clearly for pure scattering, i.e., small N_e, the ratio is independent of height in the atmosphere except for very small geometrical dilution effects. As one goes higher in the atmosphere, we should, then, obtain more nearly (49). Thus, if one insists upon using (48) he writes

$$T_e = \frac{h(\nu' - \nu)}{k \ln \frac{E_\nu}{E_{\nu'}}} \qquad (50)$$

when actually he should write, if it is fairly high in the atmosphere:

$$T_\nu = \frac{h(\nu' - \nu)}{k(\ln \frac{E_\nu}{E_{\nu'}} + 3 \ln \frac{\nu}{\nu'})} \qquad (51)$$

Thus we see that the T_e values obtained from measures in the continuum, under the assumption that the observed radiation is chromospheric emission and not scattered radiation, are likely to be erroneous. A priori, with no knowledge of T_e and N_e, it is impossible to predict the relative importance of the emission and scattering terms. Thus it seems the decision as to the utility of the method can be made best by comparing "apparent" values of T_e derived from equation (50). Since T_e presumably increases upward, and from (47) we see that the scattering increases in relative importance upward, the observed T_e will apparently decrease upward if the scattering is significant. Thus the utility of this third method is questionable, but a check on its utility is possible.

III. Conclusion

The foregoing provides an outline of the methods proposed to reduce the 1952 eclipse data. The results should provide values of N_e, T_e, ∇T_e, ∇N_e at a number of heights in the atmosphere. Hence, there are internal checks on the atmospheric structure. Thus, in view of the objective of trying to reconcile radio and optical data, one should from the foregoing have a consistent, well-defined set of results directly relevant to the fundamental problem.

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